The centrality of metaphor in the teaching of mathematics

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ABSTRACT: One of the challenges of teaching mathematics is that it is not about anything, literally. Mathematical objects (triangles, groups, surds, etc) do not exist in the real world. It is impossible to pick up a circle, although it is fairly easy to pick up a small piece of metal in the shape of a circle. The task of a (constructivist) teacher is to present to the student experiences from which the student may abstract the various aspects of the generalised mathematical CONCEPT and construct them into his own personal concept. In this paper it is argued that metaphor is the principal—perhaps the only—tool at the teacher's disposal to achieve this, and that the most important job of the teacher is to select the metaphor for presentation to the student which will most readily help her to construct her own concept. Examples will be presented to show that any given mathematical CONCEPT typically has several metaphors from which the teacher may choose, and also that no single metaphor can ever be robust enough to faithfully represent all the characteristics of the CONCEPT. The teacher therefore is faced with the task of selecting not a single metaphor, but a sequence of metaphors, the union of which will be able to represent all the characteristics of the CONCEPT.

Key-words: metaphor; teaching; mathematics

Introduction

By far the most common reason people give for disliking mathematics is that it has no meaning for them. Tasks such as solving the quadratic equation $2x^2 - x - 3 = 0$ are shrouded in mystery for many because the equation means exactly nothing at all to them. The task is quite literally meaningless. The challenge for the teacher of mathematics is to find a way or ways in which to make mathematics meaningful for children who find it incomprehensible. One particularly promising way to do this is to find ways in which people can think about mathematical concepts in terms of things they already understand, that is to find metaphors. We begin this paper by discussing briefly the cornerstone of modern mathematics learning.

1. Constructivism

It is a simple task to memorise a few words, for instance, *yat*, *yee*, *sam*, *say*, and to be able to repeat them on demand. These particular words are not arbitrary but do in fact carry a precise conventional meaning to millions of people. However without further information from an external source, no amount of introspection will allow someone who does not already know the words to derive their meaning. In the late Nineteenth Century some schools in Britain even found that it was easier to get children to memorise a book than it was to teach them to read (COULSON, 1996, for example). Learning in this way, what we might call the transmission view of learning (because teachers simply 'transmit' knowledge from their brains to those of the children), is effective in some circumstances. For instance, in getting people to learn the National Anthem we require them to learn exactly the same set of words and to sing them to the same tune at exactly the same time. It is not necessary (although it may be desirable) for the people to understand what the words mean. For instance, the words of the Singaporean National Anthem are in Malay, but only about 15% of the population of Singapore can speak Malay.

In contrast a "core belief in contemporary approaches to learning is that knowledge and cognitive strategies are actively constructed by the learner." (BEREITER, 1985, p. 201). This view denies the assumption that words can of themselves carry meaning independent of someone to interpret them. Learners are viewed as active constructors of meaning from the interaction of what they perceive with what they already know. In mathematics education this characterisation of learning is not now merely a core belief, but has become ubiquitous, although there are very many variations on the theme, from Piaget's ideas of accommodation and assimilation (PIAGET, 1972), through social constructivism (VYGOTSKY, 1980) and radical constructivism (VON GLASERSFELD, 1991, for example).

If it is the case that individuals can construct their own mathematical schemas, then, as René Thom put it "… all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics.' (THOM 1973, p. 204) and it is to this we now turn.

2. The nature of mathematics and the conceptual metaphor

Mathematics as a discipline dates back to ancient times. Its origins probably lie in analytical methods of solving practical problems. Of course, many practical problems are similar and it must have been realized very quickly that methods for solving one particular problem could be applied to similar problems, and hence abstract methods of solving generalised problems gradually grew into a systematized discipline.

Mathematics then is abstract and its objects of study are also abstract. A triangle is not a real object in the sense that it exists in our physical universe. It is not possible to pick up a triangle, but it *is* possible to pick up a piece of card which is roughly the *shape* of a triangle. So if we cannot hold a triangle in our hands, but we *can* hold one in our minds, then where does it come from? Lakoff and Nùñez have argued that "It comes from us! We create it, but it is not arbitrary… Mathematics is a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment, and our long social and cultural history." (LAKOFF and NÙÑEZ, 2000, p. 9). Their book, *Where mathematics comes from* (WMCF) is devoted to explaining how humans conceive and understand mathematical concepts. They describe how mathematical concepts—such as TRIANGLE—arise through *conceptual metaphor*, the "…mechanism by which the abstract is comprehended in terms of the concrete" (LAKOFF and NÙÑEZ, 2000, p. 5).

Richard Skemp described how one can think of mathematical concepts as a hierarchy of levels (SKEMP, 1989a, pp. 49–71). He begins with "primary" concepts; these derive directly from physical sensory experience. For instance, RED, GREEN, BLUE are primary concepts. COLOUR "is a secondary concept which is formed when we realize what the concepts [RED, GREEN, BLUE] etc. have in common." (SKEMP, 1989a, p. 56). Similarly the concept SHAPE derives from direct (visual) experience with SQUARE, CIRCLE, TRIANGLE.

When one realizes that physical objects, such as plastic counters, have both COLOUR and SHAPE, then, according to Skemp, one constructs another secondary concept, ATTRIBUTE, which is "more" abstract than the concepts from which it was generated. Thus a hierarchy of mathematical constructs can be built, layering concept upon concept, evermore abstract. Other academic disciplines can also be thought of in this way, but it is mathematics which is the example par excellence of a hierarchy of concepts.

In his discussion of the hierarchy of concepts Skemp does not ever use the term "conceptual metaphor" but it is quite clear from his description of what he called "centre our attention" that that is precisely the type of mechanism he imagined taking place when a person abstracts a concept from a set of things which all possess that concept. Since the basic task of teachers of mathematics is to help students progress up the hierarchy of concepts, then a natural tool for them is the conceptual metaphor, and we now look at an example which demonstrates the teaching implication.

3. Methods and metaphors

Some years ago the son of one of my neighbours came to my house weekly for some personal tuition in mathematics. On one occasion he turned up almost in tears, tears of frustration. He had been trying to find the values of expressions like sin 135°, tan 120º and similar functions where the angle specified was greater than 90º. His difficulty was that he had only learned trigonometric functions in the context of a right-angled triangle, as in figure 1.

In figure 1, the sine of the angle marked x is calculated by dividing 11 cm by 15 cm, symbolically

$\sin x = \frac{11}{15}$

$= 0.73333333333$

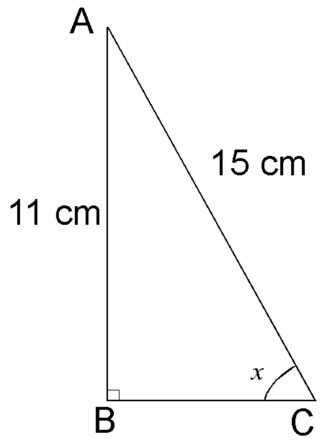

Figure 1

Adopting the conventional illustrative table (see LAKOFF and NÙÑEZ, 2000, p. 42, for example), we may represent the metaphor my young friend was using thus.

TRIGONOMETRY IS A PROPERTY OF TRIANGLES	
Source Domain: Triangles	*Target Domain*: Trigonometry
The measure of an angle in a (right angled) triangle	A value of *x*
Ratio of opposite side to hypotenuse of that triangle	The value of sin of *x*

(Of course, there are other rows which belong in this table, and all the others which follow, but in the interest of brevity we shall omit rows to which we do not need to make reference.)

Since the measure of an angle in a (right angled) triangle cannot be greater than 90º, then the value of the parameter to the sine function, within this metaphor, cannot be greater than 90º either. Neither teachers nor students usually explain what they are doing in terms of metaphors, but doing so allows us to pinpoint the root cause of my tutee's problem; he was employing an inadequate metaphor. No amount of explanation using the metaphor TRIGONOMETRY IS A PROPERTY OF TRIANGLES is likely to help, simply because this metaphor is not powerful enough to represent trigonometric functions in the richness necessary to evaluate sin 135°. We need a different metaphor; the one I employed was related to the unit circle.

Mathematically, this is simply a circle of radius 1 unit centred on the origin, O, of a pair of Cartesian coordinates.[1] We add a (counter-clockwise) rotating point P on the circumference, the radius connecting the point to the origin, and the marked angle, ϕ, which that radius makes

[1] This is a simple way of describing the unit circle which ignores the conceptual metaphors upon which it is built. For a detailed explanation, the interested reader is referred to Lakoff and Nùñez (2000, pp. 387–393).

with the *x*-axis, as in figure 2. We now *define* the value of sin ϕ to be the *y*-coordinate of P. Metaphorically speaking, we can set up the following table. (We ignore here the strict requirement to establish a correspondence between an angle and its measure, and also between the coordinates of a point and the numerical value attached to them.)

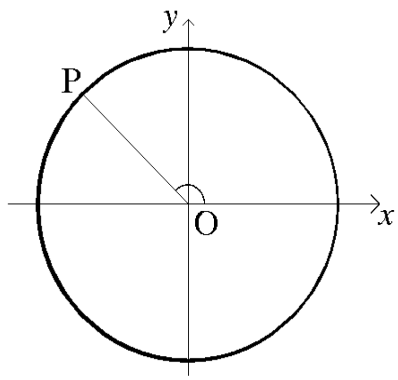

Figure 2

TRIGONOMETRY IS A PROPERTY OF THE UNIT CIRCLE	
Source Domain: Unit Circle	*Target Domain*: Trigonometry
The measure of the angle of rotation of P	A value of ϕ
The value of the *y*-coordinate of P	The value of sin of ϕ

Not only does this metaphor enable us to understand what "the value of sin 135°" means, but it also allows us to understand how the sin function can take *any* value. We can easily extend the metaphor to include cosine and tangent.

Teachers would normally think of the two metaphors, TRIGONOMETRY IS A PROPERTY OF TRIANGLES and TRIGONOMETRY IS A PROPERTY OF THE UNIT CIRCLE to be different *methods* of teaching trigonometry rather than as metaphors. In mathematics textbooks the former method is almost always chosen, and rarely with any kind of justification; presumably it is because the teachers want to be able to use the trigonometric ratios to calculate angles and lengths of sides of triangles. This is rather harder to do with the second method, one would need to go on to show how the sine and cosine functions arrived at via the unit circle can be used in right-angled triangles, probably by setting up a metaphor between the properties of triangles and the unit circle.

The teacher has to choose then between one metaphor/method which can be quickly applied to practical problems, but which cannot be extended, or another metaphor/method which is initially a little more complicated to apply, but permits a satisfactory extension to more advanced trigonometry. Similar choices confront teachers of mathematics every day. Generally, teacher training courses in mathematics education do not provide a coherent set of principles to guide trainees in these choices, but we shall argue that metaphoric analysis, such as the above for trigonometry, provide exactly the tool the teacher of mathematics needs to make choose when two or more metaphor/methods are available. The following case study of teaching about the integers shows how the tool can be used.

4. Case study: The integers

Generations of students have struggled—usually without much success—to master an understanding of the integers (…,⁻3, ⁻2, ⁻1, 0, 1, 2, …) and how to compute using them[2]. They have grappled with such odd phrases as *what is minus three minus minus seven?*, or *two negatives make a positive* (only to discover that negative three plus negative five is negative eight). And why is it that negative three times negative five has to be positive fifteen? And why don't calculators work the way we teach children? On my desk, one calculator gives 3 as the result of 2 – – 5, another gives –3, and a third won't do it all, reporting an error. By examining the various metaphors available to teachers we shall be able to see the root cause of these problems and also how the teacher can avoid them.

The negative integers are usually to be found in the curriculum for children aged about 11–12. This is probably because the negative integers are the first quantities children meet which cannot be instantiated by physical objects; one may have 16 apples, $^3/_5$ of a pie or \$0.65, but one cannot have ⁻4 cats. Children therefore need to be able to manipulate abstract concepts rather than physical embodiments. We shall return to this point later.

Most school textbooks (in English) introduce negative integers through the metaphor of ambient temperature (METCALF, 2006, p. 27 for instance), as if the correspondence between components of the domains were obvious. In fact, there is a problem here. Consider the metaphor INTEGERS ARE TEMPERATURES as in the table below.

INTEGERS ARE TEMPERATURES	
Source Domain: Temperatures	*Target Domain*: Integers
0ºC	0
Temperature above 0ºC	Positive integer
Temperature below 0ºC	Negative integer
Addition of temperatures	Addition of integers

The problem is that one of the first things we wish to do is to add integers, for instance 37 + 15 = 52. As the metaphor stands we get 37ºC + 15ºC = 52ºC, but what interpretation can we put on this? It doesn't make very much sense to add the temperature on, say, Monday (37ºC) to the temperature on Tuesday (15ºC) to get 52ºC—what can it mean? One possibility is to have addition of integers correspond to a *rise* in temperature, so 37ºC + 15ºC = 52ºC might mean "The temperature was 37ºC and it rose by 15ºC so the temperature now is 52ºC" (not that unlikely an occurrence in the UAE!)

[2] We distinguish between two terms: *minus*, conventionally symbolized by the en-dash. –, and *negative*, here symbolized by a raised hyphen, ⁻. Hence 2 – 5 is articulated "two minus five" and 2 – ⁻5, articulated as "two minus negative five".

The difficulty is that now a positive integer, say 3, corresponds to two entities in the source domain, a temperature above 3ºC *and* a rise in temperature of 3ºC. These are not the same thing at all. An analogous difficulty arises with subtraction. (The situation is worsened when we try to interpret subtraction as a fall of a negative quantity. I can model 10 – 4 by putting 10 apples into a pile and removing 4 of them. But I cannot model $10 - {}^{-}4$ in the same way.) Whilst one can imagine an adroit teacher handling this problem, when we try to include multiplication into the metaphor we begin to stretch credibility. We need to expand the metaphor.

INTEGERS ARE TEMPERATURES	
Source Domain: Temperatures	*Target Domain*: Integers
0ºC	0
Temperature above 0ºC *or* a rise in temperature	Positive integer
Temperature below 0ºC *or* a fall in temperature	Negative integer
Multiplication of temperatures	Multiplication of integers

How would we interpret 4 × 3? We cannot even consider 4ºC × 3ºC because the result would be $12ºC^2$, an impossible entity. We could manage 4 × 3ºC since that would be understood by children to mean 3ºC + 3ºC + 3ºC + 3ºC, but here the 4 is now a multiplier: a count of the number of additions, not a temperature nor a rise in temperature. We now have a third interpretation for a positive integer. The situation is getting out of hand and we need to abandon this metaphor.

Some authors have invoked other metaphors for the integers; for instance, heights above sea level and depths below. This metaphor breaks down as quickly as the temperature metaphor—multiplying a height by another height must give us an area because height is really a measure of length. INTEGERS AS MONEY is another oft-tried metaphor. It has the attraction of practicality in the source domain but many children have difficulty with the idea of lending (subtracting) a debt (a negative integer). (The recent world-wide ‘credit crunch’ might lead us to suspect many bankers have a spot of difficulty with it too.)

Few textbooks persist with the temperature, or indeed any of the others; most move swiftly to INTEGERS AS POINTS ON THE NUMBER LINE. This metaphor is quite complex and the reader is referred to Lakoff & Nùñez, (2000, p. 281) for a full account. We illustrate the spirit of the metaphor in the following diagram.

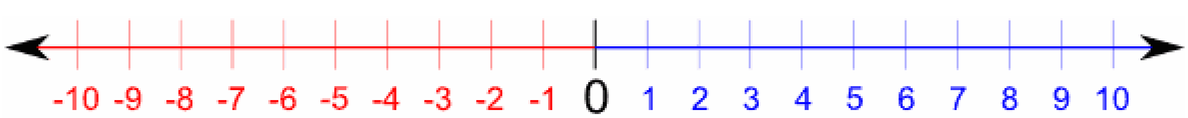

Figure 3

INTEGERS AS POINTS ON THE NUMBER LINE	
Source Domain: Point locations on a line	*Target Domain*: Integers

(The point labeled) ‘0’	0
(Labelled) points to the right of ‘0’	Positive integers
Points to the left of ‘0’	Negative integers
The point *m* is to the right of point *n*	$m > n$
The distance between *m* and *n*	The absolute difference between *m* and *n*

(There are many more rows in the complete metaphor.)

Although this gives a visual interpretation of the integers, especially of their relative sizes, the source is scarcely less abstract than the target, and it does nothing to illuminate the difficulties described at the beginning of this case study. Nevertheless, many teachers of mathematics persist with this metaphor anyway, with the subsequent failure of many children to understand. Earlier it was pointed out that negative integers are typically introduced to children aged about 11–12. Although some children will at this age have moved to Piaget’s stage of formal operations, many will still be in the stage of concrete operations. Piaget characterised this as one where children can carry out abstract operations but only with concrete things. A metaphor which exploits some kind of concrete apparatus would therefore be particularly apposite. There is such a one, INTEGERS AS COLOURED COUNTERS, and it is to this we now turn.

INTEGERS AS COLOURED COUNTERS	
Source Domain: Counters	*Target Domain*: Integers
A single green counter	1
A single red counter	$^{-}1$
m green counters	*m*, a positive integer
n red counters	*n*, a negative integer
A juxtaposition of 1 green counter and 1 red counter	0
Physically place something onto the mat	Addition
Physically take something from the mat	Subtraction

Being concrete, this metaphor requires physical activity on the part of the children. They require a mat to play on. Only counters which are physically on the mat represent anything, counters off the mat are not involved. The children need to know that *p* plus 0 is just *p* and also that *p* minus 0 is *p*. The value of the game (or answer) is the value of the counters, wherever they may lie, on the mat. So if there are 5 green counters on the mat the value is 5, and if there are 6 red counters on the mat the value is $^{-}6$. Zero is represented by a juxtaposition of 1 green counter and 1 red counter, so if there are, say, 5 green counters and 1 red counter, we may juxtapose the red counter with one of the green counters. This gives us a zero so the juxtaposed counters are removed from the mat and the value of the game is 4. At any time if an equal number of red and green counters is added to (or taken from) the mat, the value of the game is unaltered. In this metaphor addition of integers is modeled by adding things to the mat. Some examples will illustrate the method.

Question	Setup	Action	Result
3 + 5	Begin with 3 green counters on the mat	Place 5 green counters on the mat	There are now 8 green counters on the mat. The value of the game is 8.
2 + ¯5	Begin with 2 green counters on the mat.	Place 5 red counters on the mat. Match one green counter with one red counter, their combined value is zero so we can remove the pair without changing the value of the game. Repeat this manoeuvre.	There are no green counters on the mat, but there are 3 red counters left on the mat. The value of the game is ¯3.
8 – 3	Begin with 8 green counters on the mat	Remove 3 green counters from the mat	There are now 5 green counters on the mat. The value of the game is 5.
2 – 6	Begin with 2 green counters on the mat	Remove 6 green counters from the mat. We cannot do this because there are only 2 green counters there. Add 1 green counter and 1 red counter together; their value is zero so there is no change to the value of the game. There are now 3 green counters on the mat. Add another green-red pair; their value is zero so there is no change to the value of the game, but there are now 4 green counters. Add 2 more green-red pairs, there are now 6 green counters on the mat so we can remove them.	There are 4 red counters on the mat. The value of the game is ¯4.
¯3 – ¯7	Begin with 3 red counters on the mat	We need to remove 7 red counters but there are only 3 there. Add a green-red pair, their combined value is 0 so the value of the game is unchanged, but there are now 4 red counters on the mat. Add another green-red pair—there are 5 red counters on the mat. Add another green-red pair, and another. There are now 7 red counters on the mat, and 4 green ones. Remove the 7 red counters.	There are 4 green counters on the mat. The value of the game is 4.

It is not difficult to expand the table for INTEGERS AS COLOURED COUNTERS to extend the metaphor to account for the things we need to do with integers. However in the analysis table above, there is only one row for addition: in fact we need four rows, one for each of the possible additions types: positive + positive positive + negative negative + positive negative + negative

We also need four rows for subtraction, multiplication and division, but in the interests of brevity we omit them here.

There is however one problem. We cannot model the division of a positive integer by a negative integer using counters. Consider $12 \div {}^{-}4$; whether we use partitioning or measuring we have to confront the fact that we cannot form $^{-}4$ groups—the count of a collection logically must be a positive integer. With this single exception, the metaphor INTEGERS AS COLOURED COUNTERS covers everything we need to use as teachers. But should we really use counters in secondary school? And what happens when the children have to take an exam? They may be allowed calculators but they are never allowed counters. In practice it does not take children very long to move from using physical counters to pictures of counters and there is no prohibition on drawing diagrams in mathematics. The following diagram illustrates the last example in the table above.

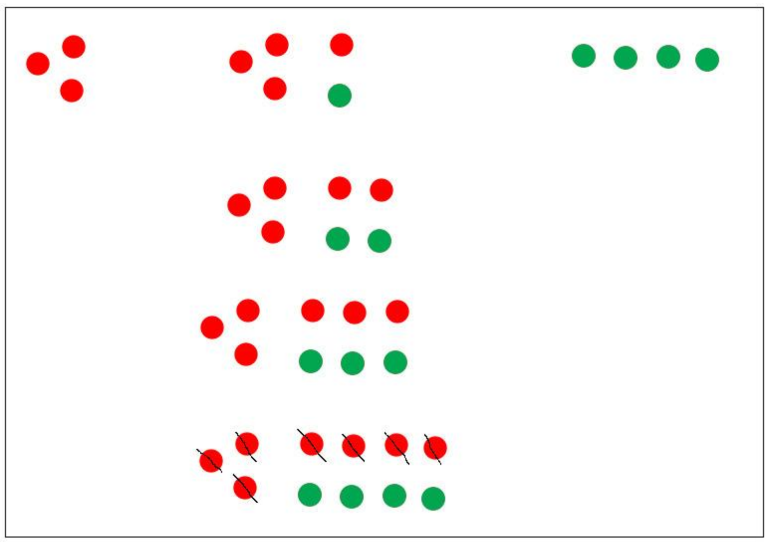

Figure 4

5. Implications for teachers

The most important corollary of metaphoric analysis for teachers is that it provides a planning tool. For some years now I have introduced student teachers to the principle of planning lessons by metaphoric analysis. In essence they consider a topic and construct analysis tables for every metaphor they can find. By laying these out side-by-side a student can compare the metaphors against each other. In the case of integers we might have

INTEGERS ARE TEMPERATURES
INTEGERS ARE ALTITUDES
INTEGERS ARE BANK BALANCES
INTEGERS ARE POINTS ON THE NUMBER LINE

INTEGERS ARE ARROWS
INTEGERS AS COLOURED COUNTERS

and more beside. By examining the analysis tables the student can see that all of the metaphors deal with zero quite well: as the freezing point of water, as sea level, as no cash in the bank, as the centre point of the number line, as the arrow with no length and a green counter and a red counter in juxtaposition. The student now may choose whichever of these s/he thinks best fits the children; the student may even choose to use all of them. However, when the student comes to plan the first lesson on addition examination of the tables will show that some of the metaphors do not allow the proper correspondences, as we saw earlier. The crucial point is that the teacher has a rational method for choosing or not choosing any particular metaphor to use in a particular lesson.

In the case of the integers, the best correspondence revealed by analysis of the tables is INTEGERS AS COLOURED COUNTERS, but this does not mean the student is obliged to use it. Often we find that the most complete metaphor is the most difficult to introduce—this should not surprise us because the better the match with the target domain, the closer the source will be to it conceptually and the more removed from real-life applications of the concept. As a consequence a teacher may well choose one metaphor for one part of a topic and a different one for another part. There is a danger lurking here though.

Textbooks, for good reasons of cost, rarely include more than one method/metaphor for a particular topic, and almost never more that two. This raises a number of issues, but one in particular is relevant here. It can happen surprisingly often that the metaphor used in the text does not sit very well with the experiences of the children, INTEGERS AS BANK BALANCES, for example. In such a situation teachers will, or at least should, draw upon their prior experience to present an alternative to the metaphor used in the text. Teachers (usually) have well-developed understandings of the mathematics they teach, so well in fact that they can switch effortlessly between metaphors. The difficulty is that their students cannot because they are still making sense of the material. There is a considerable risk of a teacher switching from one metaphor to another without explicitly pointing this out to the students. The result is a sort of pedagogical mixing of metaphors; the teacher is using one metaphor and the student is thinking with another. Teachers therefore need to take great care when switching between metaphors.

When considering which metaphor to employ for, say, the integers, some are a ‘better fit’ than others. The question naturally arises as to whether or not there is a ‘perfect fit’ that is a single metaphor which establishes a one-to-one correspondence between the source domain and the target domain. In mathematics, it is hard to see how there can be. For if there was a one-to-one correspondence between every aspect of the source and target domains, then both would possess precisely the same properties, in which case they would be the same thing but with two different names. A mathematician would probably call this situation an isomorphism and promptly lose interest.

Conclusion

Mathematics is a hierarchical discipline concerned solely with abstract concepts, built layer upon layer of metaphors. In the early years of study the metaphors can be drawn from the real world, but children do not have to learn very much mathematics before the metaphors involved are purely from one abstract domain to another. When this happens there is a danger that the teacher may consider the metaphors too difficult for the children and short-cut the process. For instance, when introducing the division of fractions, one well-known American elementary text book dispensed with any attempt to imbue any understanding of the process, instead entreating children "Ours not to reason why, just invert and multiply". But it is not necessary to by-pass an understanding of mathematics. Richard Skemp (1989b) produced a set of materials for use in primary school which although it does not mention metaphor once, is in fact a carefully structured set of activities which lead children through the layers of conceptual metaphors to a full understanding of the mathematics.

Teachers must be able to lead their children through their respective curricula as Skemp did his, by constructing a sequence of activities—mental or physical as appropriate—which take the children through the learning process. The tool of conceptual metaphoric analysis provides them with precisely the tool they need.

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Endnote: For those who are wondering, *yat*, *yee*, *sam*, *say* is *one, two, three, four* in Cantonese.

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