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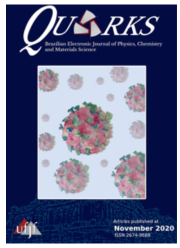

Schwinger Model as Prototype for Confined Fermions

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Abstract

We will present the Schwinger Model by characteristics like Mechanism of Higgs-Schwinger, Fermionic Charge Shielding, and Chiral Anomaly. Topological Vacuum is a prototype of theories with Confined Fermions. We will present some aspects of Bosonization and, for example, we will make a representation of the Free Field of Dirac with null mass. We will do a review of Schwinger's model with Lowestein-Swieca, and we will discuss the theory. We also will present modified models of Rothe-Stamatescu, Schroer and Thirring, demonstrating its equivalence with Sine-Gordon's theory.

Keywords: Confinement, Fermions, QCD, QED, Two-dimensional.

1. Introduction

The strong interaction has the characteristic of being a confining interaction, ie, an interaction where the simplest charge structure is a dipole (see Figure 1), and the form of the interaction is linear, so the charges always appear in pairs. The major problem is that the fermionic determinant of 4-dimensional interaction (3 + 1) is non-analytical. That fact turns the elaboration of a nonperturbative theory a very complicated task. Therefore, to infer some behaviors of the hadrons, a group of researchers, among them the Professor Jorge André Swieca, began working with two-dimensional models in quantum electrodynamics (QED).

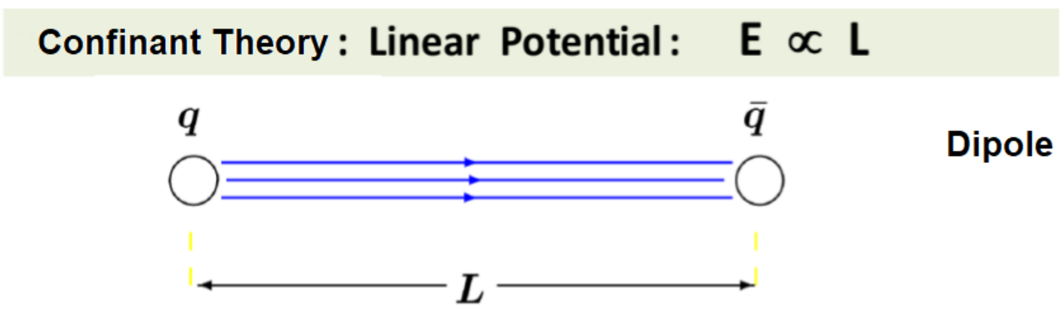

Figure 1 – Field lines in Electromagnetism with dimensionality (1 + 1).

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The Schwinger Model can be basically characterized by [1]:

a. The observable content of the theory is fully described by a massive pseudo-scalar free field (the theory is exactly soluble);

b. All physically feasible states have null global charge: charge shielding effect;

c. Fermions, which are the bearers of this shielded quantum number, are absent from the physical spectrum: confinement of fermions;

d. Theory is asymptotically free: although the fermions lose individually the observable meaning, they behave to short distances as free particles;

e. All free charge and pseudo-charge of the theory condenses in the vacuum by spurious operators, originating a complex structure for the vacuum.

Due to these characteristics, the Schwinger Model was then considered as a prototype for a Quark Confinement Model. The complex structure of the vacuum in the Schwinger Model [2] also provided support for recent studies on tunneling via instantons in non-abelian gauge theories.

In the next sections we will present the Lagrangian and study the symmetries of the Schwinger model, models with massive and massless fermions. We will also present the bosonization and the models: modified Rothe-Stamatescu, Thirring and Schroer.

2. Lagrangean Noether's Theorem and Symmetries

We will revisit the results obtained in the References [1,3-8].

The following conventions will be adopted in this work

$$g^{00} = 1;$$

$$\epsilon_{01} = -\epsilon^{10} = 1;$$

$$\gamma^{\mu}\gamma^{5} = \epsilon^{\mu\nu}\gamma_{\nu};$$

$$\gamma^{5} = \gamma^{0}\gamma^{1};$$

$$\gamma^{0} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

$$\gamma^{1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};$$

$$\gamma^{5} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Variables in light cone:

$$x^{\pm} = x^{0} \pm x^{1};$$

$$\partial_{\pm} = \partial_{0} \pm \partial_{1};$$

$$\mathcal{A}_{\pm} = \mathcal{A}_{0} \pm \mathcal{A}_{1}.$$

For a null mass free scalar field φ and the pseudo-scalar $\tilde{\varphi}$:

$$\varphi(x) = \varphi(x^+) + \varphi(x^-);$$

$$\tilde{\varphi}(x) = \varphi(x^+) - \varphi(x^-);$$

$$\tilde{\partial}_\mu = \epsilon_{\mu\nu}\partial^\nu;$$

$$\tilde{\partial}_\mu\tilde{\varphi} = -\partial_\mu\varphi.$$

The basic method of the Quantum Field Theory (QFT) to describe a given interaction is to write a Lagrangean, via Noether's Theorem, find conserved currents and a gauge transformation. In QED the most general possible Lagrangean, considering massless fermions is

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \overline{\psi}(i\gamma^\mu\partial_\mu - e\gamma^\mu\mathcal{A}_\mu)\psi, \tag{1}$$

where $\psi, \tilde{\psi}$ are spinors, also known as fermionic variables described as

$$\psi = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}, \tag{2}$$

$$\overline{\psi} = \psi^\dagger\gamma^0 = [\psi_1^*, \psi_2^*]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = [\psi_2^*, \psi_1^*], \tag{3}$$

and $\mathcal{F}^{\mu\nu} = \partial^\mu\mathcal{A}^\nu - \partial^\nu\mathcal{A}^\mu$, is the electromagnetic tensor

$$\mathcal{F}^{\mu\nu} = \begin{bmatrix}\mathcal{F}^{00} & \mathcal{F}^{01}\\ \mathcal{F}^{10} & \mathcal{F}^{11}\end{bmatrix}, \tag{4}$$

where its components are

$$\mathcal{F}^{00} = \partial^0\mathcal{A}^0 - \partial^0\mathcal{A}^0 = \mathcal{F}^{11} = \partial^1\mathcal{A}^1 - \partial^1\mathcal{A}^1 = 0;$$

$$\mathcal{F}^{01} = -\mathcal{F}^{10} = \partial^0\mathcal{A}^1 - \partial^1\mathcal{A}^0.$$

Where $\mathcal{A}_\mu = (\mathcal{A}_0, \mathcal{A}_1)$ is the vector potential. The first term of Lagrangean gives us the field of free photons $-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ the second term introduces fermions as sources of electromagnetism $\tilde{\psi}(i\gamma^\mu\partial_\mu - e\gamma^\mu\mathcal{A}_\mu)\psi$.

Explaining the components of the Lagrangean,

$$\mathcal{L} = \frac{1}{2}\mathcal{F}^{10}\mathcal{F}_{10} + \psi_2^*(i\gamma^0\partial_0 - e\gamma^0\mathcal{A}_0)\psi_1 + \psi_1^*(i\gamma^1\partial_1 - e\gamma^1\mathcal{A}_1)\psi_2. \tag{5}$$

Gauge transformations, which are transformations that preserve symmetry

$$\psi \to \psi'(x) = \psi\exp[i\phi(x) + i\gamma^5\phi(x)], \tag{6}$$

$$\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu[f(x)]. \tag{7}$$

here $\phi(x), f(x)$ are respectively a bosonic field and a function. Conserved currents, or equations of motion corresponding to existing symmetries are given by

$$\gamma^\mu(i\partial_\mu\psi(x) + e\mathcal{A}_\mu\psi(x)) = 0, \tag{8}$$

$$\partial_\mu \mathcal{F}^{\mu\nu} + ej^\nu = 0, \tag{9}$$

where $j^\mu = \overline{\psi}\gamma^\mu\psi$ is the vector current of the electromagnetism associated with symmetry $U(1)$, when this current is conserved, ie, $\partial_\mu j^\mu = 0$ the symmetry $U(1)$, is shown. When $\partial_\mu j^\mu \neq 0$ we say that the symmetry is broken, there is still the chiral symmetry $U_{quiral}(1)$ associated with the chiral current $j^{\mu 5} = \overline{\psi}\gamma^5\gamma^\mu\psi$, this symmetry is constrained to the chiral axis, which defines projectors on a given direction. Explaining the components

$$\partial_0 \mathcal{F}^{01} + ej^1 = 0; \;\; \partial_1 \mathcal{F}^{10} + ej^0 = 0, \tag{10}$$

$$j^\mu = (\partial_1 \mathcal{F}^{10}, \partial_0 \mathcal{F}^{01}). \tag{11}$$

In the next sections we will present some simple models and explore the presence of the symmetries presented here.

2.1. Free Massless Fermion Model

The model is defined by the following Lagrangean density

$$\mathcal{L} = i\overline{\psi}^0(x)\gamma^\mu\partial_\mu\psi^0, \tag{12}$$

where the spinor ψ has components

$$\psi^0 = \begin{bmatrix} \psi_1^0 \\ \psi_2^0 \end{bmatrix}, \tag{13}$$

and represents the simplest model, only with what we call the kinetic term. The equations of motion for the components are

$$i\partial_+\psi_1^0 = 0, \tag{14}$$

$$i\partial_-\psi_2^0 = 0, \tag{15}$$

Therefore

$$\psi_1^0(x) = \psi_1^0(x^-),$$
$$\psi_2^0(x) = \psi_2^0(x^+).$$

The Model presents the global chiral symmetry $U^5(1)$, whose transformations are

$$\psi' = e^{i\alpha}\psi,$$
$$\psi' = e^{i\alpha\gamma^5}\psi.$$

We can write the vector and axial currents in terms of the normal product

$$j^\mu =\vdots\, \overline{\psi}^0\gamma^\mu\psi^0 \,\vdots, \tag{16}$$

$$j^{5\mu} =\vdots\, \overline{\psi}^0\gamma^\mu\gamma^5\psi^0 \,\vdots. \tag{17}$$

The notation $\vdots\,(A)\,\vdots$ means that the current is calculated as the product of operators at the same point as the expansion of the Wilson operator for short distances [8]. As a consequence of the overall present symmetries $U(1) \otimes U^5(1)$ the currents are conserved.

$$\partial^{\mu} j_{\mu} = 0, \tag{18}$$

$$\partial^{\mu} j_{\mu}^{5} = 0. \tag{19}$$

The two-point functions of the free fermion are calculated to be [1,7]

$$\langle 0|\psi_2^{\dagger 0}(x^-)\psi_2^0(0)|0\rangle = \frac{1}{2i\pi x^+}, \tag{20}$$

$$\langle 0|\psi_1^{\dagger 0}(x^+)\psi_1^0(0)|0\rangle = \frac{1}{2i\pi x^-}. \tag{21}$$

2.2. Bosonization

The definition of *bosonization* is "Representing the fermionic operator in terms of ordered Wick exponentials of the null mass scalar field". Therefore, the null mass free scalar field plays a very important role in the bosonization of the Fermi fields in two-dimensional models with asymptotic freedom [3,9]. The fundamental block of bosonization is the free scalar field of Wick ordered exponentials with null mass, defined by:

$$:e^{i\alpha\varphi(x)}: = e^{i\alpha\varphi^-(x)}e^{i\alpha\varphi^+(x)}, \tag{22}$$

where $\varphi^{\pm}(x)$ are respectively the destruction and construction operators

$$\langle 0|\varphi^-(x) = 0;\ \varphi^+(x)|0\rangle = 0. \tag{23}$$

Although the null mass free scalar field in 2 dimensions is not a well-defined field, due to infrared divergence in the 2-point function, the corresponding ordered exponentials of Wick, see Equation (22), are well-defined operators since they satisfy the charge selection rule [6,11]

$$\langle 0|\Pi_{j=1}^{n}: e^{i\alpha_j\varphi(x_j)}:|0\rangle \neq 0; \text{ if and only if } \Sigma_j\alpha_j = 0. \tag{24}$$

The 2-point function of the null mass free scalar field

$$\langle 0|\varphi(x)\varphi(0)|0\rangle = cte\int_0^{\infty}\frac{dp}{p}e^{-ipx}, \tag{25}$$

which diverges in the infrared. To evaluate it we have to enter a cutoff μ,

$$\langle 0|\varphi(x)\varphi(0)|0\rangle = cte\int_0^{\infty}\frac{dp}{p}\left(e^{-ipx} - \theta(\mu - |p|)\right), \tag{26}$$

with the equation of motion

$$\Box\varphi(x) = 0 \rightarrow \partial_+\partial_-\varphi(x) = 0, \tag{27}$$

introducing the components

$$\varphi(x) = \varphi(x^+) + \varphi(x^-), \tag{28}$$

$$\tilde{\varphi}(x) = \varphi(x^+) - \varphi(x^-), \tag{29}$$

we have

$$\langle 0|\varphi(x\pm)\varphi(y^{\pm})|0\rangle = -\frac{1}{4\pi}\ln[i\mu(x^{\pm}-y^{\pm}+i\epsilon)], \tag{30}$$

and

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = -\frac{1}{4\pi}\ln[-\mu^2(x-y)^2], \tag{31}$$

therefore of indefinite metric. The motivation of bosonization comes from the fact that the 2-point function of the Wick exponential, Equation (22), serves

$$\langle 0|:e^{i\alpha\varphi(x)}::e^{-i\alpha\varphi(y)}:|0\rangle = e^{\alpha^2[\varphi^-(x^{\pm}),\varphi^+(y^{\pm})]}, \tag{32}$$

$$\langle 0|\psi_j(x\pm)\psi_j^{+}(y^{\pm})|0\rangle = \frac{1}{2\pi x\pm}, \tag{33}$$

where $j = 1{,}2$ The both expressions are identical if we choose $\alpha = 2\sqrt{\pi}$

$$\psi(x^{\pm}) = \sqrt{\frac{\mu}{2\pi}}:e^{2i\sqrt{\pi}\varphi(x^{\pm})}. \tag{34}$$

2.3. Free Massive Fermions

The Lagrangean Density that describes the model of the massive fermion is

$$\mathcal{L} = i\overline{\Psi}(x)\gamma^{\mu}\partial_{\mu}\Psi(x) + m\overline{\Psi}(x)\Psi(x), \tag{35}$$

where $m\overline{\Psi}(x)\Psi(x)$, is the mass term and the corresponding equations of motion are

$$i\gamma_{\mu}\partial^{\mu}\Psi + m\Psi(x) = 0. \tag{36}$$

The symmetry $U(1)$ is preserved

$$\partial^{\mu}j_{\mu}(x) = 0. \tag{37}$$

The symmetry $U^5(1)$ is explicitly broken due to the presence of the mass term

$$\partial^{\mu}j_{\mu}^{5} \neq 0. \tag{38}$$

The solution of the operators is given by the so called Mandelstam operator [8,9]

$$\Psi(x) = \sqrt{\frac{\mu}{2\pi}}:e^{i\sqrt{\pi}[\gamma^5\tilde{\varphi}(x)+\int_{x'}\partial_0\tilde{\varphi}(x)dx]}:. \tag{39}$$

The bosonized form of the mass operator gives

$$:\overline{\Psi}(x)\Psi(x):= -\frac{\mu}{\pi}:\cos(2\sqrt{\pi}\varphi(x)):. \tag{40}$$

It is interesting to note that the equivalence between the Sine-Gordon model and the massive free fermion implies the equivalence of a non-interacting fermion and an interacting boson [11].

3. Bosonization of Schwinger Model

To bosonize the Schwinger Model we will start from the equations of motion already presented [12,14,15]

$$\gamma^\mu(i\partial_\mu\psi(x) + e\mathcal{A}_\mu\psi(x)) = 0, \tag{41}$$

$$\partial_\mu\mathcal{F}^{\mu\nu} + ej^\nu = 0, \tag{42}$$

which exhibit invariance against local and global gauge transformations γ^5.

The expression of the Ansatz of the fermionic operator $\psi(x)$ in terms of the bosonic variables can be motivated, to a certain extent, by characteristics of the classical version of the theory. We consider the classical Dirac field $\psi^C(x)$ in the presence of a potential vector $\mathcal{A}_x^c$ which we can written in the general form

$$\mathcal{A}^{c\mu} = \frac{\epsilon_{\mu\nu}\partial^\mu\tilde{\Xi}}{e} + \frac{\partial_\mu f(x)}{e}, \tag{43}$$

where the first term represents the transverse component that contributes to the electric field

$$\mathcal{F}_{\mu\nu} = \frac{2}{e}\epsilon_{\mu\nu}\Box\tilde{\Xi}(x), \tag{44}$$

the second term $\partial_\mu f(x)$ is a longitudinal contribution of pure gauge. The Dirac equation is then written as

$$i\gamma^\mu\partial_\mu\psi^c(x) = -\gamma^\mu\left[\epsilon_{\mu\nu}\partial^\nu\tilde{\Xi}(x) + \partial_\mu f(x)\psi^c\right], \tag{45}$$

using $\gamma^\mu\gamma^5 = \epsilon^{\mu\nu}\gamma_\nu$ one can write

$$i\gamma^\mu\partial_\mu\psi^c(x) = \left[-\gamma^\mu\gamma^5\partial_\mu\tilde{\Xi}(x) + \partial^\mu\partial_\mu f(x)\right], \tag{46}$$

suggesting a solution of the exponential type

$$\psi^c(x) = \exp[\gamma^5\tilde{\Xi}(x) + f(x)]\psi^{(0)c}(x), \tag{47}$$

corresponding to the classical version

$$\psi_c^{(0)}(x) = \exp[ia[\gamma^5\tilde{\phi}(x) + \phi(x)]]. \tag{48}$$

Using the theory's gauge freedom, we chose the Lorentz gauge with $\Box f = 0$ so the Maxwell equations reduce to:

$$\Box\mathcal{A}_\mu^c(x) + ej_\mu^c(x) = 0, \tag{49}$$

giving now the classical current

$$j_c^\mu(x) = -\frac{\epsilon^{\mu\nu}\partial_\nu\Box\tilde{\Xi}(x)}{e}. \tag{50}$$

In this way, compatibility with the existence of a conserved nontrivial current requires $\Box\tilde{\Xi}(x) \neq 0$ and $\tilde{\Xi}$ acts as potential for $j^\mu_{c(x)}$. This, therefore, motivates us to introduce in quantum theory the bosonization Ansatz for the fermionic field $\psi(x)$ as the quantized version of the exponential solution,

$$\psi(x)=\sqrt{\frac{\mu}{2\pi}}:e^{i\sqrt{\pi}\left(\phi(x)+\gamma^5\tilde{\phi}(x)\right)}::e^{i\gamma^5\tilde{\Xi}(x)}:, \tag{51}$$

where $\phi,\tilde{\phi}$ are respectively the potential and the pseudo-potential for free current and free pseudo-current

$$j_l^\mu(x)=-\frac{1}{\sqrt{\pi}}\partial^\mu\phi(x)=\epsilon^{\mu\nu}\tilde{J}_{l\nu}\,, \tag{52}$$

and $\tilde{\Xi}(x)$ is an independent free field, the specific nature of which shall be determined at a later stage, requiring compatibility with the equations of motion. The fact that the $\tilde{\Xi}$ field is independent of $\phi(x),\tilde{\phi}(x)$ guarantees anticomutativity for $\psi(x)$.

In dealing with a quantum theory, we can no longer use the formal equation of motion, since it involves the product of field operators considered at the same point. We then introduce the prescription of the normal product of operators

$$i\gamma^\mu\partial_\mu\psi(x)=-\frac{1}{2}e\lim_{\epsilon\to 0,\epsilon^2<0}\gamma^\mu\left[\mathcal{A}_\mu(x+\epsilon)\psi(x)+\psi(x)\mathcal{A}_\mu(x+\epsilon)\right]. \tag{53}$$

The application of the Dirac operator to the fermionic field $\psi(x)$ results

$$i\gamma^\mu\partial_\mu\psi(x)=\gamma_\nu\epsilon^{\mu\nu}[\partial\tilde{\Xi}^-(x)]\psi(x)+\psi\left(x\left[\partial_\mu\tilde{\Xi}^+\right]\right), \tag{54}$$

where we made use of the fact that

$$\gamma^\mu\partial_\mu\psi^0(x)=0, \tag{55}$$

we rewrite it as a limit

$$i\gamma^\mu\partial_\mu\psi(x)=\frac{1}{2}\lim_{\epsilon\to 0,\epsilon^2<0}\gamma_\mu\epsilon^{\mu\nu}\left[\partial_\mu\tilde{\Xi}^-(x+\epsilon)\psi(x)+\psi(x)\partial\tilde{\Xi}^+(x+\epsilon)+\partial_\mu\tilde{\Xi}^-(x-\epsilon)\psi(x)\right. \\ \left.+\,\psi(x)\partial_\mu\tilde{\Xi}^+(x-\epsilon)\right], \tag{56}$$

inserting the null commutator

$$[\partial_\mu\tilde{\Xi}(x+\epsilon),\psi(x)]+[\psi(x),\partial_\mu\tilde{\Xi}(x-\epsilon)], \tag{57}$$

$$i\gamma^\mu\partial_\mu\psi(x)=\frac{1}{2}\lim_{\epsilon\to 0,\epsilon^2<0}\gamma_\nu\epsilon^{\mu\nu}\left[\partial_\nu\tilde{\Xi}(x+\epsilon)\psi(x)+\psi(x)\partial_\mu\tilde{\Xi}(x-\epsilon)\right], \tag{58}$$

$$i\gamma^\mu\partial_\mu\psi(x)=-\frac{e}{2}\lim_{\epsilon\to 0\epsilon^2<0}\gamma^\mu[\mathcal{A}_\nu(x+\epsilon)\psi(x)+\psi(x)\mathcal{A}_\nu(x-\epsilon)], \tag{59}$$

we identify the gauge field $\mathcal{A}_\nu(x-\epsilon)$

$$\mathcal{A}_\mu=-\frac{1}{e}\epsilon_{\mu\nu}\partial^\nu\tilde{\Xi}(x). \tag{60}$$

To calculate the current j^μ we introduce a limit procedure by way of example of what we have done above

$$j^{\mu}(x)=\lim_{\epsilon\to 0,\epsilon^2<0} Z^{-1}(\epsilon)\Big[e^{-ie\int_x^{x+\epsilon}\mathcal{A}_{\mu}^{-}(z)dz^{\mu}}\overline{\psi}(x+\epsilon)\gamma^{\mu}\psi(x)e^{-ie\int_x^{x+\epsilon}\mathcal{A}_{\mu}^{+}(z)dz^{\mu}} - \langle 0|\overline{\psi}(x+\epsilon)\gamma^{\mu}\psi(x)|0\rangle\Big], \tag{61}$$

the singularities are eliminated by subtracting the expected value in the vacuun's current and by the normalization factor Z^{-1}. Substituting Equations (51) and (60) in (61), we obtain

$$j^{\mu}(x)=-\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_{\nu}\phi(x)-\frac{1}{\pi}\epsilon^{\mu\nu}\partial_{\nu}\tilde{\Xi}(x), \tag{62}$$

and thus we identify the expressions for the basic objects of the theory in terms of the bosonic field $\tilde{\Xi}(x)$. In order to verify its structure we consider Maxwell's equations in the Lorentz gauge

$$\Box\mathcal{A}_{\mu}(x)+ej_{\mu}(x)=0, \tag{63}$$

replacing Eqs. (60) and (62), we find the equation of motion in terms of the scalar potential $\tilde{\Xi}$

$$(\Box+\frac{e^2}{\pi})\Box\tilde{\Xi}(x)=0, \tag{64}$$

whose general solution can be written in terms of two other scalar fields

$$\tilde{\Xi}(x)=\alpha\tilde{\eta}(x)+\beta\tilde{\Sigma}(x), \tag{65}$$

where $\tilde{\Sigma}(x)$ is massive and $\tilde{\eta}(x)$ is not massive

$$(\Box+\frac{e^2}{\pi})\tilde{\Sigma}(x)=0, \tag{66}$$

$$\Box\tilde{\eta}(x)=0. \tag{67}$$

Therefore, the objects $\psi(x),\mathcal{A}_{\mu}(x),j^{\mu}$ assume the following expressions

$$\psi(x)=\sqrt{\frac{\mu}{2\pi}}:e^{i\sqrt{\pi}[\phi(x)+\gamma^5\tilde{\phi}(x)]+i\gamma^5[\alpha\tilde{\eta}(x)+\beta\tilde{\Sigma}(x)]}:, \tag{68}$$

$$\mathcal{A}_{\mu}(x)=-\frac{1}{e}[\alpha\partial_{\nu}\eta(x)+\beta\epsilon_{\mu\nu}\partial^{\nu}\tilde{\Sigma}(x)], \tag{69}$$

$$j^{\mu}(x)=-\frac{1}{\sqrt{\pi}}\partial_{\nu}\left[\phi(x)+\frac{\alpha}{\sqrt{\pi}}\eta(x)\right]-\frac{\beta}{\pi}\epsilon_{\mu\nu}\partial^{\nu}\tilde{\Sigma}(x), \tag{70}$$

$$\epsilon_{\mu\nu}\partial^{\nu}\tilde{\eta}(x)=\partial_{\nu}\eta(x). \tag{71}$$

Given that we assumed the field $\tilde{\Xi}$ independent of the fields $\phi(x),\tilde{\phi}(x)$ results then in the longitudinal contribution of null mass to the current

$$L_{\mu}(x)=-\frac{1}{\sqrt{\pi}}\partial_{\mu}\left[\phi(x)+\frac{\alpha}{\sqrt{\pi}}\eta(x)\right], \tag{72}$$

this particular gauge does not provide the realization of operators of the Maxwell equations. However, by a procedure similar to that employed by Gupta-Bleuler in the four-dimensional version of the theory we can introduce the Hilbert space H with scalar product defined in such a way that we can construct a subspace $\widetilde{H}$ with positive metrics at which the expected value of the operator $L_\mu(x)$ is null. In this way the Maxwell equations are satisfied, among physical states belonging to $\widetilde{H}$.

For physical states belonging to $\widetilde{H}$ we must require

$$\left\langle \theta_i \left| \partial_\mu [\phi(x) + \frac{\alpha}{\sqrt{\pi}\eta(x)}] \right| \eta_j \right\rangle = 0, \nu(\theta)|0\rangle \in \widetilde{H}, \tag{73}$$

that is, the subspace $\widetilde{H} \subset H$ is generated by applying to the vacuum of operators of a certain class defined via

$$[\theta(x), L_\mu(x)] = 0, \forall(x, y). \tag{74}$$

Introducing, then, the total Hilbert space H as an indefinite scalar product, by quantizing in negative metric for the field $\tilde{\eta}(x)$, that is, according to

$$\langle 0|\tilde{n}(x)\tilde{\eta}(y)|\rangle = (-)\tfrac{1}{i}\Delta^+(x-y;0), \tag{75}$$

and parameterizing $\alpha = \sqrt{\pi}$, we ensure the operator

$$L_\mu(x) = -\frac{1}{\sqrt{\pi}}\partial_\mu[\phi(x) + \eta(x)], \tag{76}$$

belongs to class $[\theta]$

$$[L_\mu(x), L_\nu(y)] = 0, \tag{77}$$

creating from the vacuum states of null norm

$$||\partial_\mu(\eta + \phi)|0\rangle||^2 = 0. \tag{78}$$

In this way, the Maxwell equations are satisfied between physical states belonging to the gauge invariant subspace $\widetilde{H}$

$$\langle\theta|\partial_\nu \mathcal{F}^{\mu\nu}(x) + ej^\mu(x)|\theta\rangle, \tag{79}$$

which by construction, has positive norm states

$$\langle\theta|\theta\rangle \geq 0. \tag{80}$$

One can note that the fermions field $\psi(x)$ does not create physical states, since it does not commute with the longitudinal current. However, we can include in the class $[\theta]$ of operators the new fermionic field obtained through gauge transformation of operators, that is

$$\Psi(x) = e^{i\gamma R^-(x)}\psi(x)e^{i\gamma R^+(x)}, \tag{81}$$

where $\gamma = \sqrt{\pi}, R(x) = \eta(x)$ and therefore

$$\Psi(x) = \sqrt{\frac{\mu}{2\pi}} : e^{i\sqrt{\pi}[\eta(y)+\phi(x)+\gamma^5(\tilde{\eta}(x)+\tilde{\phi}(x))]} : : e^{i\beta\gamma^5\tilde{\Sigma}(x)} :, \tag{82}$$

provides a solution to the Dirac equation, with the corresponding gauge field given by

$$\mathrm{A}(x) = -\frac{\beta}{e}\epsilon_{\mu\nu}\partial^\nu\tilde{\Sigma}(x). \tag{83}$$

One can observe that the transverse component of this field A_μ

$$\mathrm{A}_\mu(x) = \frac{\pi}{e}[j_\mu(x) - L_\mu(x)], \tag{84}$$

obeys the equation of motion for the free massive field

$$(\Box + \mu_0^2)\mathrm{A}_\mu(x) = 0, \tag{85}$$

with $\mu_0^2 = \frac{e^2}{\pi}$ indicating that the gauge field acquires a physical mass different from zero (Schwinger's Mechanism). In dealing with a free field, the constant β can be adjusted so that $\mathrm{A}(x)$ follows the canonical rules of commutation for a massive vector field.

$$[\mathrm{A}_\mu(x), \mathrm{A}_\nu(y)] = -i(\frac{\beta\mu_0}{e})^2\left[g_{\mu\nu} + \frac{1}{\mu_0^2}\partial_\mu\partial_\nu\right]\Delta(x-y;\mu_0), \tag{86}$$

thus suggesting the parameterization $\beta = \sqrt{\pi}$ which as we shall see later assures for the theory an asymptotic free behavior at short distances.

We have thus seen that the gauge invariant subspace $\tilde{H}$ with positive scalar product, can be explicitly generated by the vacuum application of the operator class θ, which is constructed from $[\mathcal{F}_{\mu\nu}, j_\mu, \Psi(x), L_\mu(x)]$ and Wightman polynomials and operators composed of these.

Given that the operator $L_\mu(x)$, when applied to any physical state, generates a null norm state, we can then define a new subspace Q, with the positive scalar product defined, consisting of equivalence classes, in which operator $L_\mu(x)$ is identically null and the equations of motion are used as identity between operators.

Let $H^0 \subset H$ be the subspace of $\tilde{H}$ constituted by all states $[|\theta^0\rangle]$ with null norm in $\tilde{H}$, that is, states obtained by applying at least one operator $L_\mu(x)$ to any state of $\tilde{H}$. The two states $|\theta_n\rangle, |\theta_m\rangle \in \tilde{H}$ will be equivalent, that is

$$|\theta_n\rangle = |\theta_m\rangle, (\mathrm{mod} H^0), \tag{87}$$

$$|\theta_n\rangle - |\theta_m\rangle \in H^0, \tag{88}$$

then we can consider equivalence classes

$$[|\theta\rangle] = [|\theta\rangle - |\theta^0\rangle], \tag{89}$$

which will constitute the quotient space $Q = \tilde{H}/H^0$ of defined positive metric and in which

$$[|\theta^0\rangle] \simeq 0 \text{ em } Q. \tag{90}$$

Therefore, since

$$\left[L_\mu(x), \theta(y)\right] \equiv 0 \forall (x-y) \text{ e } \forall \theta |0\rangle \in \widetilde{H}, \tag{91}$$

then Q, in the space of the operators of equivalence classes, operator $L_\mu(x)$ will be identically null, thus result in the realization of operators of the Maxwell's equations.

In a gauge theory, all the physical content must be contained in the set of invariant operators, we must therefore always consider the space of the observable states (physical subspace) constructed by applying invariant gauge operators to the vacuum.

It may be seen then that states which have no definite positive norm are devoid of observable meaning in the sense that the objects of theory which are responsible for the appearance of these states become spurious excitations which bear no physical significance.

The null mass scalar fields (assumed Goldstone bosons) are eliminated, and the structures of the gauge invariant objects, such as the current $j^\mu(x)$ and the electric field $\mathcal{F}_{\mu\nu}(x)$, are fully written in terms of the free massive pseudo-scalar field $\tilde{\Sigma}(x)$.

Formally we can still compose bilocal operators corresponding to observables as

$$D(x,y) \doteq \psi(x) e^{ie\int_x^y A_\mu dz^\mu} \psi^\dagger(y), \tag{92}$$

where $A_\mu(x)$ and $\psi(x)$ are given respectively by (82), (83). The symbol $\doteq$ in (92) indicates an operator product, which needs to be defined correctly in terms of the normal product of the exponentials of the bosonic fields involved and that will be timely realized later.

Before, however, we will consider some representation features for the gauge field algebra $\psi(x)$ where

$$\psi(x) = (\frac{\mu}{2\pi})^{\frac{1}{2}} \colon e^{i\sqrt{\pi}[\eta(x)+\phi(x)] + i\sqrt{\pi}\gamma^5[\tilde{\eta}(x)+\tilde{\phi}(x)+\tilde{\Sigma}(x)]} \colon, \tag{93}$$

which satisfies the subsidiary condition (77), by creating physical states belonging to $\widetilde{H}$. Rewriting

$$\psi(x) = (\frac{\mu}{2\pi})^{\frac{1}{2}} \colon e^{i\sqrt{\pi}\gamma^5\tilde{\Sigma}(x)} \colon \sigma(x), \tag{94}$$

where the operator $\sigma(x)$ carries all the free charge and free pseudo-charge of the theory, that is,

$$[Q_l, \sigma(x)] = -\sigma(x), \tag{95}$$

$$\left[\tilde{Q}_l, \sigma(x)\right] = -\gamma^5 \sigma, \tag{96}$$

is due to quantization with negative metric for the field $\tilde{\eta}$, represents a constant operator, which does not generate physical excitations, condenses in the vacuum producing degenerate vacuums; reflecting in this way the spontaneous breaking of the global and gauge symmetries and γ^5 without the appearance of Goldstone bosons.

To visualize, we consider for simplicity the two-point function for the fields $\psi(x)$,

$$\langle 0 | \psi^\dagger(x) \psi(y) | 0 \rangle = \left(\frac{\mu}{2\pi}\right) e^{\pi \gamma_x^5 \gamma_y^5 \Delta^+(x-y,\mu_0)} \langle 0 | \sigma^\dagger(x) \sigma(y) | 0 \rangle, \tag{97}$$

where the correlation function for the fields $\sigma(x)$ is easily calculated and results

$$\langle 0 | \sigma_1^\dagger(x) \sigma_1(y) | 0 \rangle = \langle 0 | e^{-2i\sqrt{\pi}[\eta^+(u)+\phi^+(u)]} e^{2i\sqrt{\pi}[\eta^-(u\prime)+\phi^-(u\prime)]} | 0 \rangle =$$

$$= e^{4\pi[\eta^+(u),\eta^-(u\prime)]+4\pi[\phi^+(u),\phi^-(u\prime)]} = 1, \qquad (98)$$

$$\langle 0|\sigma_2^\dagger(x)\sigma_2(y)|0\rangle = e^{4\pi[\eta^+(v),\eta^-(u')]+4\pi[\phi^+(u),\phi^-(v')]} = 1, \qquad (99)$$

since the commutators involved have opposite signals due to quantization with negative metric for the field $\eta(x)$.

We therefore have for the functions of Wightman $2n$ points

$$\langle 0|\Pi_{k=1}^n \psi(x_k)\Pi_{j=1}^n \psi^\dagger(y_i)|0\rangle =$$

$$\left[\Pi_{k=1}^n e^{\pi\left[\gamma_{xk}^5\gamma_{xj}^5\Delta^+(x_k - x_j\mu_0)+\gamma_{yk}^5\gamma_{yj}^5\Delta^+(y_k-y_j\mu_0)\right]}\Pi_{k,j}e^{\pi\gamma_{xk}^5\gamma_{yj}^5}\Delta^+\left(x_k - y_j\mu_0\right)\right]W[\sigma], \qquad (100)$$

where $W[\sigma]$ is the function of $2n$ points for the fields σ_i

$$W[\sigma] = \langle 0|\Pi_{j=1}^k \sigma(x_j)\Pi_{l=k+1}^n \sigma(x_l)\Pi_{\gamma=1}^k \Pi_{m=k+1}^n \sigma^\dagger|0\rangle = 1. \qquad (101)$$

Since $\sigma_1(x)$ and $\sigma_2(x)$ are two independent fields, i.e.,

$$\sigma_1(x) \equiv \sigma_1(u), \sigma_2(x) \equiv \sigma_2(u),$$

results then

$$[\sigma_1(x), \sigma_2(y)] \equiv 0, \forall(x-y), \qquad (102)$$

and, due to the combination of fields $(\eta + \phi)$ create null norm states from vacuum, it follows that

$$[\sigma_1(x), \sigma_1(y)] \equiv [\sigma_2(x), \sigma_2(y)] \equiv 0, \forall(x-y). \qquad (103)$$

From the fact that σ_1 fields give rise to constant Wightmann functions, that is, they do not interact with each other, results by translational invariance

$$\sigma_i = \sigma_i(0)\sigma_i\,, \qquad (104)$$

and consequently

$$\sigma_i^*\sigma_i(0) = \sigma_i\,. \qquad (105)$$

Thus, σ_i objects constitute a class of constant unit operators, that just cannot be reduced to identity because they carry selection rules.

This all results in a violation of property, whereby when two points become infinitely separated by a spacelike distance, then the correlation between the fields at these points disappears. For the two points function (97) the cluster property requires

$$\lim_{a\to\infty}\langle 0\left|\psi^\dagger(x)\psi(x+\lambda a)\right|0\rangle = \langle 0|\psi(x)|0\rangle\langle 0|\psi|0\rangle, \qquad (106)$$

and as free charge and pseudo free charge are conserved, we have, to the right side

$$|\langle 0|\psi(x)|0\rangle|^2 = 0. \qquad (107)$$

However, by explicitly calculating the limit of (97), we obtain

$$\lim_{a\to\infty}\langle 0\left|\psi^{\dagger}(x)\psi(x+a\lambda)\right|0\rangle = \frac{\mu}{2\pi}, \tag{108}$$

thereby violating cluster decomposition. This then reveals that the Schwinger model vacuum is not unique. Generally speaking, cluster decomposition (for Green functions associated with charge-carrying invariant gauge operators and non-zero chirality) is violated, a consequence of the fact that operators factor into a dynamic part, which only depends on the massive field $\tilde{\Sigma}(x)$, and a spurious part consisting of fields σ_i, which condenses in vacuum all the free charges and free pseudo charges of the theory.

$$\tilde{\theta} = \theta^{din}(x) \times \theta^{spurious}. \tag{109}$$

Resulting

$$\lim_{a\to\infty}\langle 0|\theta^{\dagger}(x)\theta(x+a\lambda)|0\rangle = \langle 0|\theta^{\dagger}(x)|n\rangle\langle n|\theta(x)|0\rangle \neq |\langle 0|\theta(x)|0\rangle|^2 = 0, \tag{110}$$

where $|n,m\rangle$ belongs to a set of degenerate vacuums, which are characterized by the phenomenon of condensation of spurious operators.

$$|n,m\rangle = \sigma^n\sigma^m|0\rangle. \tag{111}$$

This infinite degeneration of the vacuum then characterizes the spontaneous breaking of the global symmetry of gauge and γ^5 without the emergence of Goldstone bosons.

To restore cluster decomposition, we introduce the parameterized vacuum in terms of an angle $\theta \in [0,2\pi]$ through superposition

$$|\theta\rangle = |\theta_1\theta_2\rangle = \frac{1}{2\pi}\sigma_{n_1n_2}^{\infty} =_{-\infty} e^{-i(\eta_1\theta_2+\eta_2\theta_2)}|n_1n_2\rangle, \tag{112}$$

such that

$$\sigma_j|\theta\rangle = e^{i\theta_i}|\theta\rangle. \tag{113}$$

Thus, we have the explicit breaking of the corresponding symmetries, that is,

$$\lim_{a\to\infty}\langle\theta|\theta^{\dagger}(x)\theta(x+a\lambda)|\theta\rangle = |\langle|\theta|\rangle|^2 = cte, \tag{114}$$

and cluster decomposition is then restored.

With "physical gauge" $\gamma = \sqrt{\pi}$ the bosonized expression for the fermionic field operator takes the form

$$\psi(x) = (\frac{\mu}{2\pi})^{\frac{1}{2}} : e^{i\sqrt{\pi}\gamma^5\tilde{\Sigma}}(x) : \begin{bmatrix} e^{i\theta_1} \\ e^{i\theta_2} \end{bmatrix}, \tag{115}$$

where we explicitly identify σ_1, σ_2 with the phases $e^{i\theta_1}, e^{i\theta_2}$.

After discussing some relevant aspects of representation for the gauge field algebra $\psi(x)$ and revealing the complex vacuum structure of the Schwinger model.

We now consider the construction of bilocal objects $D(x,y) = \psi(x)e^{ie\int_x^y A_\mu dz^\mu}\tilde{\psi}(x)$ using Equation (93) for fermionic field operator $\psi(x)$, we correctly introduce the bosonized expression for the bilocal operator in terms of the normal product of the involved exponentials, such as

$$D(x,y) = N(x-y){:}\, e^{i\sqrt{\pi}[\gamma_x^5 \tilde{\Sigma}(z) - \int_x^y \epsilon_{\mu\nu}\partial^\nu \tilde{\Sigma}(z) dz^\mu - \gamma^5 \tilde{\Sigma}(y)]}{:}, \quad (116)$$

where the normalization matrix

$$N(x) = -\frac{1}{2\pi}\begin{bmatrix} \dfrac{-i}{x_0 + x_1} & \mu\sigma_1\sigma^* \\ \mu\sigma_2\sigma_1^* & -\dfrac{i}{x_0 - x_1} \end{bmatrix}, \quad (117)$$

to ensure that the gauge invariant compound operators of the theory are generated from point separation boundaries, such as

$$j^\mu(x) = -\lim_{\epsilon\to 0} Tr[\gamma^0\gamma^\mu d(x+\epsilon, x)] = -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu \tilde{\Sigma}(x), \quad (118)$$

$$\tilde{j}^\mu(x) = \lim_{\epsilon\to 0} Tr[\gamma^0\gamma^\mu d(x+\epsilon, x)] = \frac{-1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu \tilde{\Sigma}(x), \quad (119)$$

where $d(x+\epsilon, x) = D(x+\epsilon, x) - \langle 0|D(x+\epsilon, x)|0\rangle$ and although the bilocal $D(x,y)$ transform like bilinear objects, inside spin fields $1/2$, [1]. It follows that the invariant gauge algebra generated by the electric field $F_{\mu\nu}$ and by bilocal $D(x,y)$ is fully described in terms of the massive pseudo-scalar field $\tilde{\Sigma}(x)$. Therefore, as an observable, the Hamiltonian must necessarily assume the bosonized expression.

$$H = \int_\infty^\infty \frac{1}{2}{:}\,[\partial_0\tilde{\Sigma}(x)]^2 + [\partial_1\tilde{\Sigma}]^2 + \mu_0^2[\tilde{\Sigma}]^2{:}\, dx_1, \quad (120)$$

reflecting in this way the complete isomorphism between gauge invariant objects algebra of Schwinger model and the massive free meson theory.

4. The Modified Rothe-Stamatescu Model

In this section we shall introduce the Modified Rothe-Stamatescu Model [3,5,12,14-16]. The Rothe-Stamatescu Model is defined by the following Lagrangean density

$$\mathcal{L} = \overline{\psi}(x)\big(i\gamma^\mu\partial_\mu\big)\psi(x) + \partial_\mu\tilde{\phi}(x)\partial^\mu\tilde{\phi}(x) + \frac{1}{2}m_0^2\tilde{\phi}^2(x) + \mathrm{g}\big(\overline{\psi}(x)\gamma^\mu\gamma^5\psi(x)\big)\partial_\mu\tilde{\phi}(x). \quad (121)$$

Let's consider what we call the Modified Rothe-Stamatescu Model (MRSm), we do $m_0 = 0$ in Equation (19).

The term $\overline{\psi}(x)\gamma^\mu\gamma^5\psi(x)\partial_\mu\tilde{\phi}(x)$ It is called axial current coupling with the pseudo-scalar field derivative. It is the derivative coupling term. The equations of motion are:

$$\big(i\gamma^\mu\partial_\mu\big)\psi(x) = g\gamma^\mu\gamma^5\psi(x)\partial_\mu\tilde{\phi}(x), \quad (122)$$

$$\Box\tilde{\phi}(x) = -\partial_\mu \,\vdots\, (\overline{\psi}(x)\gamma^\mu\gamma^5\psi(x)) \,\vdots. \quad (123)$$

The Equations (20) and (21) suggest the Solution of operators of the equation of motion, which is given in terms of Wick's ordered exponentials,

$$\psi(x) =: e^{ig\gamma^5\tilde{\phi}(x)}: \psi^0(x), \tag{124}$$

where ψ^0 is the massless free fermionic field solution, that satisfies the Equation (2).

Due to symmetry $U(1) \otimes U^5(1)$ we have conservation of axial current (21).

$$\Box\phi(x) = 0. \tag{125}$$

Writing explicitly the fermionic field ψ^0, bosonized in terms of the field operator

$$\psi^{(0)}(x) = \left(\frac{\mu}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{\pi}{4}\gamma^5}: e^{i\sqrt{\pi}[\gamma^5\tilde{\varphi}(x)+\varphi(x)]}:, \tag{126}$$

where μ is the remnant infrared regulator of a massless free field.

The classic gauge symmetry of the first kind is preserved. We compute the vector current using the near-point prescription [5]

$$\begin{aligned} J^\mu(x) &=\vdots\, \overline{\psi}(x)\gamma^\mu\psi(x) \,\vdots \\ &= \mathcal{Z}_\psi \lim_{\epsilon\to\infty} \left[\overline{\psi}(x+\epsilon)\gamma^\mu e^{-ig\int_x^{x+\epsilon}\epsilon_{\mu\nu}\partial^\nu\tilde{\phi}(z)dz^\mu}\psi(x) - VEV\right] \end{aligned} \quad , \tag{127}$$

where

$$\mathcal{Z}_\psi = e^{g^2[\tilde{\phi}^+(x+\epsilon),\tilde{\phi}^-(x)]}. \tag{128}$$

The bosonized vector current is given by

$$J^\mu(x) = j_f^\mu(x) - \frac{g}{\pi}\epsilon^{\mu\nu}\partial_\nu\tilde{\phi}(x), \tag{129}$$

where the free fermionic current is

$$j_f^\mu(x) =\vdots \left(\overline{\psi}^{(0)}(x)\gamma^\mu\psi^{(0)}\right) \vdots= -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\tilde{\varphi}(x), \tag{130}$$

and the axial current is

$$J_\mu^5(x) = \epsilon^{\mu\nu}J^\nu(x) = -\partial_\mu\left(\frac{1}{\sqrt{\pi}}\tilde{\varphi}(x) + \frac{g}{\pi}\tilde{\phi}(x)\right). \tag{131}$$

Due to symmetry $U(1) \otimes U^5(1)$ we have the conserved currents

$$\partial^\mu J_\mu^5 = 0, \tag{132}$$

$$\partial^\mu J_\mu = 0, \tag{133}$$

which implies wave equations for the scalar fields

$$\Box\tilde{\varphi} = \Box\tilde{\phi} = 0. \tag{134}$$

5. Thirring Model whith Massive Fermion

The Thirring Model [3,9,12,14-16], with massive fermion is defined by the following Lagrangian Density

$$\mathcal{L} = i\overline{\Psi}(x)\gamma^{\mu}\partial_{\mu}\Psi(x) - m\overline{\Psi}(x)\Psi(x) + \frac{g^2}{2}J_{\mu}(x)J^{\mu}(x), \tag{135}$$

with the conserved current

$$J_{\mu}(x) =\vdots\, \overline{\Psi}(x)\gamma_{\mu}\Psi(x) \,\vdots, \tag{136}$$

and with the corresponding equation of motion

$$i\gamma^{\mu}\partial_{\mu}\Psi(x) + m\Psi(x) + gJ^{\mu}(x)\gamma_{\mu}\Psi(x) = 0. \tag{137}$$

In the massive Thirring model we have the symmetry $U(1)$, and non invariance under chiral transformations. The prescription of bosonization in the Mandelstam representation is given by:

$$\Psi_{Th}(x,t) = \sqrt{\frac{\mu}{2\pi}}e^{i\frac{\pi}{4}\gamma^5}\text{:exp}[i[\frac{\beta}{2}\gamma^5\tilde{\phi}(x,t) + \frac{2\pi}{\beta}\int_x^{\infty}\partial_0\,\tilde{\phi}(z,t)dz]]\text{:}, \tag{138}$$

where the phase $e^{i\frac{\pi}{4}\gamma^5}$ was introduced by Rothe and Swieca [17], to correctly get the mass term in the quantum equation of motion for fermions.

The constant β is related to the coupling constant of Thirring with the constant g by

$$\beta^2 = 4\pi(1 - \frac{g^2}{\pi})^{-1}, \tag{139}$$

when

$$g^2 < \pi. \tag{140}$$

In the free case $g = 0$ we have

$$\beta = 2\sqrt{\pi}, \tag{141}$$

analogously to the previous cases we find the expression of massive free fermion, Equation (18), the currents are given by [11,17]

$$J^{\mu}_{Th}(x) = -\frac{\beta}{\sqrt{2}\pi}\epsilon^{\mu\nu}\partial_{\nu}\tilde{\phi}(x), \tag{142}$$

$$J^{\mu 5}_{Th}(x) = -\frac{\overline{\beta}}{\sqrt{2}\pi}\partial^{\mu}\tilde{\phi}(x), \tag{143}$$

From Equation (45) stems, like symmetry $U^5(1)$ is broken by the mass term

$$\Box\tilde{\phi}(x) \neq 0. \tag{144}$$

To address the dynamics of this problem we will bosonize the Lagrangean, so we define the mass term of the bosonized fermion explicitly as a Sine Gordon term [3].

$$:\overline{\Psi}(x)\Psi(x) := \frac{\overline{\alpha}}{\beta^2}:\cos(\beta\tilde{\phi}(x)):. \tag{145}$$

The dynamics of the field $\tilde{\phi}$ is described by Sine-Gordon Lagrangean

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\tilde{\phi}(x))^2 + \frac{\overline{\alpha}}{\beta^2}:\cos(\beta\tilde{\phi}(x)):. \tag{146}$$

Starting from the bosonized Lagrangean we can write the equation of motion

$$\Box\tilde{\phi}(x) + :\sin[\beta\tilde{\phi}(x)] := 0. \tag{147}$$

This is the famous equivalence of Thirring's model with Sine-Gordon theory established by Coleman [11] and Swieca [6]. In the Ref. [6] a non-perturbative demonstration is presented, while [11] uses perturbative methods. So we have the well-defined quantum theory for the massive Thirring model.

6. The Schroer Model

The Schroer Model is defined by the following Lagrangean density

$$\mathcal{L} = \overline{\Psi}(i\gamma^\mu\partial_\mu)\Psi + \frac{1}{2}\partial_\mu\tilde{\xi}\partial^\mu\tilde{\xi} + g(\overline{\Psi}\gamma^\mu\Psi)\partial_\mu\tilde{\xi} + m\overline{\Psi}\Psi. \tag{148}$$

Matching the equations of motion

$$(i\gamma^\mu\partial_\mu)\Psi = g\gamma^\mu N\Psi\partial_\mu\tilde{\xi} + m\Psi(x) = 0, \tag{149}$$

$$\Box\xi = -g\partial_\mu :(\overline{\Psi}\gamma^\mu\Psi) := 0. \tag{150}$$

The Schroer's operator solution is

$$\Psi(x) =: e^{iq\xi(x)}: \Psi^0(x):, \tag{151}$$

where

$$\Psi^{(0)}(x) = (\frac{\mu}{2\pi})^{\frac{1}{2}} e^{-\frac{\pi}{4}\gamma^5} : e^{i\sqrt{\pi}\gamma^5[\gamma^5\tilde{\xi} + \int_1^\infty \partial_0\tilde{\xi}(x^0,z^1)dz^1]}:. \tag{152}$$

Therefrom,

$$\partial_\mu J^{\mu 5} \neq 0, \tag{153}$$

that is, the axial current is not conserved due to the presence of the mass term. This characterizes non-invariance with chiral transformations. $U^5(1)$. However, the Equation (49) guarantees the charge symmetry $U(1)$.

8. Conclusions

We saw that 2d Electrodynamics has fermionic confinement properties, just like QCD in 4d. What makes the Schwinger Model quite attractive as Toy Model for the study of the mysterious Strong interactions. In particular the quark confinement, which is still today one of the open questions in the Standard Model of Elementary Particles.

As a curiosity, we also presented the Rothe-Stamatescu, Thirring and Schroer Models. We also showed some properties as the equivalence of the Thirring Model with the Sine-Gordon Model.

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