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ABSTRACTIONISM AND PHYSICAL QUANTITITES

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ABSTRACT: In this paper, I present two crucial problems for Wolff's metaphysics of quantities: 1) The structural identification problem and 2) the Pythagorean problem. The former is the problem of uniquely defining a general algebraic structure for all quantities; the latter is the problem of distinguishing physical quantitative structure from mathematical quantities. While Wolff seems to have a consistent and elegant solution to the first problem, the second problem may put in jeopardy his metaphysical view on quantities as spaces. After drawing a parallelism between Wolff's treatment of quantitative structures and Frege's conception of quantitative domain, I propose a solution to the Pythagorean problem based on the idea that mathematical structures are the result of applying an abstraction principle on physical quantitative structures. In particular, I propose the view that abstraction may be seen as the operation of structure determination which transforms concrete physical quantities (i.e. undetermined structures) into abstract mathematical quantities (fully determined structures of thin and shallow objects).

Key-words: Metaphysics of Quantities, Abstraction Principles, Locationism.

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1 Introduction

Quantitative expressions represent an indispensable portion of scientific language. At a preliminary level, we have the intuition that scientific discourse is mainly about special features of observable reality which are commonly labeled as 'quantities' (e.g. mass, momentum, length, area, volume, temperature, etc...). The metaphysics of quantities is mainly concerned with the ontological status of putative denotations of quantitative attributes and their relations. Yet to frame such a large family of philosophical issues it is crucial to understand what is a quantity.

If quantities are to be identified with a special class of properties, it must be understood what is the characteristic mark of quantitative properties and in which respect they are distinct from qualitative ones. The first feature which undoubtedly seems to be ascribable to quantitative properties is a certain *plurality of specifications*. The fact that a certain object has mass seems to entail that it has a certain specific mass-property among many others which are mutually exclusive. All of these specific mass properties seem to be systematically comparable; for any two distinct mass properties m_1 and m_2 it is always possible to say whether objects having m_1 are "more massive" than objects having m_2 or vice versa. It also seems possible to make assertions about the "distance" between m_1 and m_2, i.e. to say how much m_1 (or m_2) is associated with a greater mass than m_2 (m_1). This suggests that quantities constitute metric structures and not just series of ordered properties.

Another important fact, especially if we consider how quantitative expressions are used in scientific language, is the relation between quantities and numbers. Intuitively, only quantitative attributes may be associated with numerical values and measured. Thus it seems that another distinctive feature of quantities is related to the possibility of representing their physical structures into mathematical structures. In other words, it is peculiar of quantities that they may be represented by numbers.

Regarding the structure of quantities, some interesting results of *the representational theory of measurement* (LUCE et al., 1971-1990) are illuminating. Different physical quantities have different ways of being represented in a mathematical setting. The basic idea here is that given a physical relational structure Σ and a mathematical structure M, the possibility of representing Σ in M is defined in terms of the existence of a homomorphism

Número XXVI - Volume I - julho de 2023
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between Σ and M. If we speak of quantitative properties instead of objects instantiating such properties, then we must have an isomorphism.

Clearly, there are many possible isomorphisms between the structure of a physical quantity and the structure of, e.g., real numbers. Each particular isomorphism may be seen as a particular attribution of numerical values according to a certain *scale*. The variety of physical quantities is defined considering both the particular relations which are considered as part of the quantitative structure (e.g. order, concatenation, etc...) and the invariant under scale transformation. According to the first criterion of classification, some quantities are said to be additive while others are not; according to the second, some quantities are said to be measurable on a *ratio scale*, for ratios between values of different magnitudes of the same quantity are invariant under scale transformation. By the same token, we have measurability on *interval scales* or *logaritmic scales*.

Such a variety of quantitative structures defined according to aforementioned classificatory criteria puts us in the following predicament: *how may we define a unique set of structural features as necessary and sufficient conditions for saying that a certain structure is indeed a quantity?* Perhaps 'quantity' is not the sort of concept which may be defined according to a unique set of common features, perhaps our understanding of 'quantity' is more based on a class of family resemblances rather than on a single characteristic mark. However, it seems at least undesirable that the concept of quantity – which indubitably has a crucial role in the foundation of mathematical rigor of the scientific discourse – presents such a form of vagueness. Call the problem of defining the structural features common to all quantities *the structural identification problem*.

There is an interesting approach – quite common in the metaphysics of science – to overcome this sort of problems. Whenever our taxonomical criteria fail to univocally capture a significant concept (such as 'quantity') it is worth doubting that we are "carving nature at its joints". In other words, the difficulty to define a concept according to our classificatory criteria may be due to the fact that we are using the wrong classification. In our case, the difficulty of univocally defining the concept of quantity may be due to the fact that we are looking at the wrong features of quantitative structures: perhaps the concept of quantity should be defined neither in terms of the relations constituting the structure (e.g. order, concatenation, etc...) nor of the scale type in which is numerically represented (e.g. ratio scale, interval scale, logarithmic scale, etc...). As Wolff (2020) suggests – by appealing to some interesting theorems of representational theory of measurement – what is common to all quantities is neither a certain kind of relations nor the particular numerical representation, yet the structure formed by all automorphisms on the given structure (more on this later). For this (and others) reason, in this paper we adopt Wolff's proposal as background theory.

The main consequence of such a "paradigm-shift" in the metaphysics of quantities is that we must review our conception of the relations between physical and mathematical structures. Physical quantities are not to be defined according to their relations with numbers: what makes 'mass' a physical quantity is not the fact that there allegedly exists an isomorphism (homomorphsim) between mass properties (massive objects) and the system of real numbers, yet an intrinsic feature of the system of mass properties (masses), i.e. a particular structure defined by the class of all automorphisms on the system of mass properties (masses). As a consequence, mathematical structures lose their foundational role in the definition of the concept of quantity and turn out to be a special sort of quantitative structures among others. This takes us to a second philosophical issue: if the system of real numbers is a quantitative structure among others, what is the relation between physical and mathematical quantities? What is the peculiarity of mathematical structures? If there is no fundamental difference between a continuous physical quantity (if there is any) and the system of real numbers, what prevents us from regarding all physical quantities as mathematical structures or, vice versa, all mathematical structures as physical? Call the problem of distinguishing between mathematical and physical quantitative structures – in the light of Wolff's solution to the structural identification problem – *the Pythagorean problem*. In the following it will be made clearer why we should consider the definition of this distinction as a philosophical problem.

The purpose of the present paper is to offer a solution to the Pythagorean problem starting from Wolff's solution to the structural identification problem based on the representational theory of measurement. And this task comes in the form of an aporetic trade off: if we accept Wolff's universal characterization of quantities in terms of the structure of automorphisms, then it seems that we are forced to admit that mathematical structures are quantities on a par with physical structures. On the other hand, if we are looking for a sharp distinction between physical and mathematical structures *qua* quantities, it seems that we have to reject Wolff's nice and elegant solution to the structural identification problem.

The present paper relies on two main ideas. The first idea consists in accepting Wolff's solution to the structural identification problem and trace the distinction between physical and mathematical quantitative structures in terms of the distinction between abstract and concrete structures. Thus I will argue that there are abstract and concrete quantities, and that while physical quantities are concrete, mathematical quantitative structures are abstract. The second idea consists in using the notion of abstraction principle to trace the distinction between concrete and abstract structures.

Before disputing the details of my proposal, it is worth saying few words on abstraction principles. An abstraction principle is a sort of implicit definition taking the following form: for every a, b in a certain domain of objects,

$$a \approx b \Leftrightarrow f(a) = f(b) \quad \textbf{(AP)}$$

where '$\approx$' is an equivalence relation, and 'f' is the so called *abstraction operator*. The significance of abstraction principles is commonly attributed to Frege who in §64 of his *Grundlagen* (FREGE, 1950) uses the following example to elucidate the structure of this sort of definitions: for every straight lines a and b,

$$a \parallel b \Leftrightarrow \text{dir}(a) = \text{dir}(b) \quad \textbf{(DIR)}$$

which says that a and b are parallel iff the direction of a is identical to the direction of b.

The second idea of this paper is framed in terms of Linnebo (2018)'s conception of abstract objects as defined through abstraction principles, which is applied to quantitative structures (more on this later). I will show that mathematical structures may be conceived as abstracted from other quantitative structures. Moreover, I will show that the abstraction operation implies a structural transformation: all non-trivial automorphisms in the initial structure are "transformed" into the trivial automorphism in the structure obtained by abstraction. This will provide an interesting criterion to define concrete quantitative structures: physical quantities are quantitative structures which have non-trivial automorphisms, while mathematical quantities are structures admitting only trivial automorphisms and obtained by abstraction on physical quantities .

The exposition of my proposal is organized in five main sections. In section 2 I

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will present Wolff's solution to the structural identification problem and his metaphysical theory of quantities based on this idea. In section 3 I will expose how Frege's approach to the definition of real numbers may be interpreted as an abstractionist conception of mathematical quantities which parallels Wolff's idea in many respects. In section 4 I will propose my formal theory of abstraction principles applied to structures and I will formulate my criteria for abstractness and concreteness of quantities.

2 Wolff on quantities

2.1 Wolff's solution to the structural identification problem

The first step to understand the theoretical background of the present proposal is the formulation of Wolff's solution to the structural identification problem. We have seen that physical quantities comes in a broad variety, some of them are additive (e.g. length) while others may not be (e.g. relativistic mass), some of them are numerically representable using ratio scales (e.g. mass), some others require interval scales (e.g. non-absolute temperature). Thus it is natural to ask in virtue of what we ascribe "quantitativeness" to all of them.

To answer this question it is necessary to recall some fundamental concepts of the representational theory of measurement. Let $\mathcal{A}$ be the physical structure associated with a certain quantity and defined as follows:

$$\mathcal{A} = \langle A, R_1, ..., R_n \rangle \qquad (1)$$

where A is a domain of objects and $R_1, ..., R_n$ are some unspecified relations on A. For the moment we will focus just on the structural aspects of physical quantities, thus it will not be necessary to specify which sort of objects are the elements of A or which sort of relations are $R_1, ..., R_n$.

Consider now a numerical structure $\mathcal{R}$ of positive real numbers:

$$\mathcal{R} = \langle \mathbb{R}^+, S_1, ..., S_n \rangle \qquad (2)$$

Número XXVI - Volume I - julho de 2023
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where $S_1, ..., S_n$ are numerical relations on $\mathbb{R}^+$. Every legitimate attribution of a numeric value to the elements of A may be defined as a function $\phi : A \to \mathbb{R}^+$, such that ϕ is an isomorphism between $\mathcal{A}$ and $\mathcal{R}$. At this point two clarifying remarks are needed. Firstly, if ϕ is an isomorphism between $\mathcal{A}$ and $\mathcal{R}$, then it is a one-to-one function from A to $\mathbb{R}^+$; as a consequence, we are implying that there is an uncountable infinity of physical entities – whatever they might be. This is a quite strong ontological commitment. For the moment I will not discuss this problem which will be dealt with later on in this section. Secondly, every isomorphism ϕ between $\mathcal{A}$ and $\mathcal{R}$ may be conceived as a particular scale of measurement in which the quantity having structure $\mathcal{A}$ may be measured.

Physical quantities have two interesting structural properties: 1) there is a plurality (possibly an infinity) of scales of measurement, i.e. of isomorphisms between the quantitative structure and the mathematical structure ; 2) For every two isomorphisms ϕ and ψ, there is a scale transformation τ such that $\psi = \tau \circ \phi$ with τ automorphism of the mathematical structure. In simpler terms, if every isomorphism is a unit of measurement, every transformation is a conversion from one unit to another. Now for every two isomorphisms ϕ and ψ, and for every transformation τ, it is easy to prove that the function

$$\mu = \phi \circ \tau \circ \psi^{-1} \qquad (3)$$

is an automorphism on $\mathcal{A}$. In particular, if we chose τ such that $\psi \neq \phi \circ \tau$, then μ is an non-trivial automorphism on $\mathcal{A}$, i.e. an automorphism which is not the identity function. These simple remarks show that physical quantities are characterized by a large class of non-trivial automorphisms. We may spell out this characteristic in clearer and more intuitive terms. The fact that nothing forces us to attribute a specific number to a certain magnitude of a given quantity may be interpreted as the fact that the quantity under consideration presents a *form of structural indeterminacy*. In other words, every objects x in the quantitative structure bears certain relation to others, yet there is no fact of the matter regarding its exact position in the structure: we may attribute a number r to x, or any other number, provided that we operate consistent scaling of any other numerical attribution. Magnitudes of a given quantity may be located at any position in the real number line. Such a possibility of translating the entire structure of a physical quantity through the real line corresponds to the algebraic property of having infinite non-trivial automorphisms.

Número XXVI - Volume I - julho de 2023
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I have described the property of having non-trivial automorphisms with reference to the possibility of consistently attributing different numbers to the same magnitude of a given physical quantity. However, such an appeal to measurement is not needed; it is more fruitful for our purpose to understand the existence of non-trivial automorphisms by examining the structure itself. *The existence of non-trivial automorphisms in a structure $\mathcal{S}$ amounts to the structural indeterminacy of elements of $\mathcal{S}$, i.e. to the impossibility of univocally determine an element of $\mathcal{S}$ according to its structural features in $\mathcal{S}$.*

If, on the one hand, the existence of automorphisms is a property that we may ascribe to quantities, on the other hand, it is not sufficient to pick out only quantitative structures; automorphisms are quite common across algebraic structures. Hence, the next step in the Wolffian definition of quantity is to specify – as a necessary and sufficient condition – which sort of structure the class of all automorphisms of a given structure $\mathcal{S}$ must form in order for $\mathcal{S}$ to be a quantity. Here the aforementioned paradigm-shift in the definition of quantity should be evident: in seeking a definition of the concept of quantity, we are no more looking at the structure formed by the metric relations between magnitudes of the same quantity (the quantitative structure), yet at the structure formed by the non-trivial automorphisms of the quantitative structure.

Before formulating Wolff's definition an important remark on quantitative structures is needed. We have seen that these structures present a certain amount of indeterminacy insofar as they have non-trivial automorphisms. However, the degree of indeterminacy is limited by the fact that these structures must still be representable in a perspicuous way by means of numbers. An unrestricted association between objects in our domain and numbers is not enough to ground quantitativeness, for it is quite arbitrary. The representational theory of measurement introduces an interesting concept to characterize such a degree of determinacy of quantitative structures, and thus the notion of associating numbers to objects in a "perspicuous way". Therefore, before considering Wolff's paradigm shift from basic structures to automorphisms, we have to clarify some aspects of how quantities are represented as numbers. This is precisely what Wolff does before introducing his definition. Consider just the domain A of our structure $\mathcal{A}$: it has no structure whatsoever, i.e. any possible permutation on A is an automorphism on A. Suppose that A is equinumerous to $\mathbb{R}^+$; any one-to-one function from A to $\mathbb{R}^+$ is an isomorphism, a way of representing the elements of A as real numbers, even though an uninteresting way. As a consequence, any

Número XXVI - Volume I - julho de 2023
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permutation on $\mathbb{R}^+$ is a transformation of unit of measurement (i.e. a scale) of the elements of A. At this point, we want to mathematically characterize the degree of arbitrariness in this sort of numerical representation. Notice that for every $k \in \mathbb{N}$, and for every $x_1, ..., x_k \in A$, there are always two scales τ and σ (i.e. two permutations on $\mathbb{R}^+$) such that:

$$\tau(x_1) = \sigma(x_1) \;\&\; ... \;\&\; \tau(x_k) = \sigma(x_k) \;\&\; \tau \neq \sigma \qquad (4)$$

Consider now a case in which we know that there is no arbitrariness in attributing numbers to objects. Let $\mathcal{B}$ be a structure which is representable on a ratio-scale, such as mass or length. We know that in this case all scales have the form of a multiplication by a positive real number (i.e. the scale factor). In this case it is no more the case that for every $k \in \mathbb{N}$ and for every k elements of B there are always two distinct scales agreeing on their values. Consider two distinct scales $\tau(x) = \alpha \cdot x$ and $\sigma(x) = \beta \cdot x$; for any $u \in B$,

$$\tau(u) = \sigma(u) \Leftrightarrow \alpha \cdot u = \beta \cdot u \Leftrightarrow \alpha = \beta \Leftrightarrow \tau = \sigma \qquad (5)$$

In this case, when two scales agree on one value they must be identical. In representational theory of measurement we say that the group of scales is *one point unique*, i.e. the specification of one "point" is sufficient to determine a unique scale. This concept may be generalized: for instance, interval scales are two points unique, for to pick out a unique scale, the specification of two "points" is required. Consider now the case of the set A with no structure: (4) says that for every $k \in \mathbb{N}$, A is not k points unique. Thus we have a first requirement for quantitativeness: our structure must be numerically representable in such a way that it is finitely point unique, i.e. there is $n \in \mathbb{N}/\{0\}$ such that the scale group is n points unique. This fact helps us to introduce a limitation in the degree of indeterminacy of quantitative structures; as Wolff says:

> "Abstracting away from concrete measurement procedures and particular numerical representations made visible the common group theoretical structure in virtue of which certain types of numerical representation are possible. This suggests that what makes for quantitativeness is not ratios, or 'numbers in the world', but the determinacy of certain types of structure. The automorphisms of a structure characterize its determinacy, because they show how much symmetry there is in a structure. Structures that make for

É: Revista Ética e Filosofia Política

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> quantitativeness are sufficiently determinate to be finitely point unique, while not being so rigid as to permit only the identity automorphism. " (WOLFF, 2020, p.108)

This remark is crucial to understand Wolff's approach to quantities: quantitativeness is not ascribed in virtue of certain relation of the basic structure, nor in virtue of numerical representability; it is a structural property of the group of automorphisms. However, Wolff's quoted passage is still framed in terms of representational theory of measurement. At this point Wolff makes an interesting theoretical move: it uses representational theory of measurement as a sort of Wittgenstein's ladder to understand quantitativeness which is "thrown away" after doing its formal work.

Wolff's definition relies on a special class of non-trivial automorphisms: translations. Translations are defined as automorphisms with no fixed points; with reference to our structure $\mathcal{A}$, a translation on $\mathcal{A}$ is an automorphism f on $\mathcal{A}$ such that for all $x \in A$, $f(x) \neq x$. An important theorem of representational theory of measurement (LUCE et al., 1971-1990) asserts that a structure whose translations form an ordered archimedean group is representable on real numbers with a scale group which is finitely point unique. This fact seems to capture the idea of quantities –or at least of continuous quantities – as neither completely indeterminate structures nor as completely determinate ones. If a structure has a group of automorphic translations, then it is not completely determined, for, as we have seen, non-trivial automorphism may be conceived as the marks of indeterminacy of structures; on the other hand, if such a group of automorphic translations displays enough structure (i.e. order and Archimedean property), then is finitely point unique, i.e. its numerical representation are not completely arbitrary and thus it displays a certain degree of structural determinacy. This is Wolff's solution to the structural identification problem of quantities: *a certain structure is a quantity iff it has automorphic translations which form an ordered Archimedean group*. In the next subsection we will see how this definition is used as a base to frame Wolff's metaphysical views on quantities.

É: Revista
Ética e
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2.2 Wolff's metaphysics of quantities and the Pythagorean problem

Wolff's definition of the concept of quantity is just a formal definition which says nothing on what sort of entities quantities are. Yet Wolff develops is metaphysical view on quantities exploiting some fundamental points of his algebraic definition.

Wolff's metaphysical view is a form of *structural realism* about quantities. Quantities exist and exist precisely as certain sort of structures. This means that quantities are to be identified neither with certain kinds of objects (i.e. massive objects, long objects, etc...) nor with a certain kind of properties, yet with a certain kind of structures. What matters for quantitativeness is just the group of automorphic translations. This leads to a fundamental question: if quantities are relational structures what sort of entities are their relata?

Wolff rejects three main views about the elements of quantitative structures:

1. The elements of quantitative structures *are not Platonic universals* (MUNDY, 1987). This view requires the existence of uninstantiated properties (e.g. the property of having a mass greater than the mass of the universe) which seems to be too ontological demanding. Quantitative structures are at the same time actual entities and yet do not require the existence of uninstantiated properties;

2. The elements of quantitative structures *are not Aristotelian universals* (ARMSTRONG, 1988). Wolff rejects this view on the ground that the class of instantiated universals associated with a certain quantity is not rich enough to exhibit the structure required for quantitativeness;

3. The elements of quantitative structures *are not space-time points* (FIELD, 1980); this view requires a complete distribution of quantitative attributes on space-time which seems to be an assumption which imposes a too strong requirement on the physical world;

Wolff proposes a view sympathetic with that of Arntenius and Dorr (2012) and Cowling (2014) according to which *quantities are spaces and the elements of quantitative structures are points of those spaces*. This view may be called *locationism*: for an object

Número XXVI - Volume I - julho de 2023
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ISSN: 1414-3917
e-ISSN: 2448-2137

to have a certain mass is to be located at a certain point in the mass space. Wolff stresses a crucial metaphysical aspect of his position: quantities are not spaces in a metaphorical and representational sense, according to which quantities are representable in algebraic structures which are mathematical metric spaces; quantities are themselves special sort of spaces, i.e. along with space-time there are other physical spaces, one for each fundamental physical quantity.

The identification of quantities with spaces leads Wolff to draw an interesting parallelism between the ontology of space-time and the ontology of quantities. Views such as substantivalism and relationism are now applicable also to quantities. Substantivalism about quantities is a form of realism about points in quantitative spaces, while relationism is the view that there are no points but only relations among physical objects instantiating a certain quantity. Wolff is a substantivalist, for given the richness of the quantitative structure introduced to solve the structural identification problem, he cannot require the physical world to display a complete instantiation of quantities. However, Wolff is a *sophisticated substantivalist* (POOLEY, 2005). To explain this view we need to recall the main structural features of quantities. We have seen that elements of quantitative structures are not completely determined by the structure itself, by virtue of the existence of a group of non trivial automorphisms. At this point Wolff seems to implicitly rely on the following argument: if points in quantitative space have no extra-structural features, i.e. if they have no property which is not defined by their quantitative structure, and given that quantitative structures do not completely determine their elements, then there are no conditions of complete identification of these points, not even non-descriptive conditions: in other words, *the points in quantitative spaces have no haeccetistic identities*. The impossibility of identifying points in quantitative spaces is not just an epistemic or linguistic impossibility, yet a metaphysical impossibility.

In the light of these remarks, Wolff's metaphysical conception of quantity may be conceived as the combination of three metaphysical positions: structural realism, locationism, and sophisticated substantivalism. Hence, at the heart of Wolff's metaphysical proposal lies the conception of quantitative structures as spaces. To my mind, the crucial issue of this conception is the understanding of quantitative structures as spaces in a non-mathematical sense; for if we lack a criterion for distinguishing abstract mathematical spaces from concrete physical spaces, then the distinction between representability in

Número XXVI - Volume I - julho de 2023
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mathematical spaces and identification with physical space gets blurred. Suppose that $\mathcal{A}$ is quantitative structure isomorphic to the mathematical structure $\mathcal{R}$. Wolff is willing to say that the elements of A are distinct from the elements of $\mathbb{R}^+$, for the former are "points" and the latter are "real numbers". However, a criterion for tracing this distinction does not seem to be possible in the framework of Wolff's metaphysical proposal. To which property of points of qualitative spaces we shall appeal in order to say that they are not real numbers? All facts about the points of $\mathcal{A}$ are structural facts which – given the isomorphism between $\mathcal{A}$ and $\mathcal{R}$ – are the same facts about real numbers conceived as elements in $\mathcal{R}$. To say that the relations in $\mathcal{A}$ are distinct from – albeit isomorphic to – the relations in $\mathcal{R}$ is to beg the question, for distinctness between relations is distinctness between their relata. And this is precisely how the Pythagorean problema shows up in Wolff's proposal.

One may attempt the following reply. Real numbers are defined in a more complex structure, i.e. the field of real numbers. Now this field has no non-trivial automorphisms, i.e. the field of real numbers is a completely determined structure, it is – so to say – zero point unique. Thus the structure of reals is not a quantity, and real numbers are completely defined by structural features which are distinct from the structural features of the elements of $\mathcal{A}$. This is an interesting theoretical move. However, it is not clear in what sense we are obliged to define the conditions of identity for real numbers looking at a field and not to another structure they form. Positive reals with addition and order form a quantitative structure which must be distinguishable from any other physical quantitative structure.

The Pythagoren problem – i.e. the problem of tracing the distinction between physical quantitative spaces and mathematical structures – may affect the intelligibility of Wolff's identification of quantities with spaces and, without a solution to this problem, Wolff's theory may not succeed in justifying the fundamental claim that quantities are not just representable in spaces, they are spaces. My purpose is to feel this gap appealing to a special relation between the structure of real numbers and the structure of physical quantities: abstraction. In particular, abstraction principles are expected to provide a criterion of identity for real numbers and a justification for the complete determinacy of their structure which will serve as a criterion of distinctness between the concept of real number and the concept of point in a quantitative space. To make this move, I have to clarify in what sense real numbers are "abstracted from" quantitative spaces; this is precisely what Frege – even though with different purposes and terminology – attempts to do in the last (incomplete)

Número XXVI - Volume I - julho de 2023
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part of his *Grundgesetze* (FREGE, 2013). Therefore, we need a brief detour on Frege's theory of reals and more contemporary interpretations of abstractionism.

3 Frege on quantities and real numbers

In Part III of his *magnum opus*, Frege (2013, vol.II, pp.69-243) extends the logicist project to the domain of real numbers. As in the case of natural numbers, Frege's purpose is to frame a definition of the concept of real number in purely logical terms, which according to his conception of logic include also set theoretical constructions. The idea of a general definition of real numbers was not unexplored territory at the time: Dedekind had already published his essay in which the strategy of defining "cuts" – i.e. binary partitioning – on the domain of rational numbers proved to be successful. Moreover, Dedekind's definition seems to be expressible in Frege's logical language, thus Frege might just have "translated" Dedekind's cuts in his concept-script and the whole job would have been done. And yet Frege does not follow this path. This because Frege's logicism does not consist just in logical constructions aimed at recovering the mathematical language inside the concept-script. There is an eminently philosophical ingredient in Frege's approach which cannot be reduced to the technical task of manipulating the formula language. According to Frege, any philosophical attempt to provide solid foundations for arithmetic and analysis must elucidate an essential aspects of mathematics: its universal applicability. In the case of the theory of natural numbers this goal is achieved by showing that cardinality is a logical property of (sortal) concepts. In the case of real numbers, the applicability has to take into account measurement of magnitudes. Thus the definition of real numbers must stem from the fact that they are used as numerical attributions to magnitudes of the same quantity, somehow capturing – in a mathematical fashion – the structure of quantity. Therefore, Frege's logicist project – when concerned with real analysis – is a project of explaining how the definition of the concept of real numbers is based on the logical structure of quantities. For this reason the concept of quantity is central in the introductory remarks that immediately precede the formal theory.

Frege was no pioneer in formally studying the concept of quantity (take, for instance, (HÖLDER, 1901) which Frege knew quite well); however, his approach is completely

É: Revista
Ética e
Filosofia Política

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e-ISSN: 2448-2137

different from that of his contemporaries. After having discussed how the main accounts of ‘quantity’ are circular or vague, in §161 of (FREGE, 2013), Frege exposes the reason for these difficulties in understanding the concept of quantity (he uses ‘quantity’ and ‘magnitude’ as synonyms):

> “The reason for these failures lies in asking the wrong question. There are many different kinds of magnitudes: lengths, angles, periods of time, masses, temperatures, etc..., and it will scarcely be possible to say how objects that belong to these kinds of magnitude differ from objects that do not belong to any kind of magnitude. [...] Instead of asking which properties an object must have in order to be a magnitude, one needs to ask: how must a concept be constituted in order for its extension to be a domain of magnitudes?” (FREGE, 2013, p.158)

The main problem with traditional approaches is that there is no common algebraic structure for all quantities; moreover, summing masses is a quite distinct operation from summing lengths. In other words, there is no common criterion based only on the relations of the basic structure of each quantity that may distinguish quantities from qualities. Another problem – which Frege does not mention – is that the existence of a concatenation operation corresponding to ‘sum’ on a given quantity does not seem to be a logical property, yet the subject of an empirical investigation (BATITSKY, 2002). For all these reasons, the concept which may be fruitfully formalized is not that of quantity, yet the more general concept of “domain of magnitude” (*Grössengebiet*). Since I am using the word ‘quantity’ where Frege uses ‘magnitude’, I will stick to Dummett (1991)’s translation of ‘Grössengebiet’ with ‘quantitative domain’.

What is a quantitative domain and why it should do a better work than the idea of quantity in introducing real numbers? The first thing to notice is that a quantitative domain is a set of permutations. The basic elements of the quantitative structure Frege is trying to capture are neither individuals nor properties, but a special sort of one-to-one functions, i.e. permutations. The main problem is that Frege is absolutely silent on the physical meaning of the permutations constituting a quantitative domain. Frege introduces several requirements on quantitative domains: permutations must form an ordered Abelian semi-group with respect to the operation of functional composition; the ordering relation

Revista
Ética e
Filosofia Política

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

between permutations (defined in terms of the composition operation) must be dense and Dedekind complete. The structure engendered by a quantitative domain may be easily extended to the case of a group; Frege, for some unclear reason, is interested just in absolute continuous quantities, yet we need not follow him in this kind of details. Thus a quantitative domain is a set of permutations forming an Abelian ordered group with a continuous order. Considering Frege's quoted remark, this is the subject of study for the foundations of real analysis; in other words, real numbers must be defined from the concept of quantitative domain. As I will show, in shifting his attention from quantities to quantitative domains Frege was on something big, in spite of the incompleteness of his conceptual analysis.

Dummett (1991, pp.281–283) criticizes Frege for not being general, or better, for being mysteriously specific in saying that the elements of a quantitative domain must be permutations. Hölder (1901) – roughly in the same years – was studying quantities from a more abstract perspective, working out general algebraic structures whose elements may be of any sort; Frege may have done the same. I believe that Dummett's criticism is a bit unfair. Frege was quite uncomfortable with the introduction of algebraic operations without sharp and non-contextual definitions: talking of addition or product – as we do in abstract algebra – without saying how these relations are defined and on which category of entities they apply would be unacceptable from the perspective of his logicist program. On the other hand, functional composition is an operation that may be defined using only logical vocabulary. Yet there is another aspect in using quantitative domains instead of structures of individuals which is more profound. Surely Frege noticed that there is no common algebraic structure for all quantities; this caused him to abandon the idea that what I have called the structural identification problem may be solved looking at the basic structure of each quantity. But now there is a surprising fact which may do justice to Frege's alternative approach to quantities: *quantitative domains may be interpreted as the group of automorphic translations of a given quantity; and from Wolff's definition of quantity based on representational theory of measurement, we already know that the structure of this group of permutations is the mark of quantitativeness.*. Attributing to Frege Wolff's solution to the structural identification problem would be excessive; however, at least from a formal perspective, Frege was on the right track and there is a perfect parallelism between his and Wolff's approach. Hence we may discuss the relations between quantities and real numbers – and the Pythagorean problem for Wolffian quantities – in the framework of both

Revista
Ética e
Filosofia Política

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

Frege's definitional strategy and posterior interpretations of abstraction principles.

The next step is to consider the Fregean definition of real numbers by abstraction. Part III of the *Grundgesetze* is incomplete and it ends with the proofs of the formal properties of quantitative domains; thus there is no direct textual evidence on how Frege would have proceeded from this point. However, there is an unanimous consensus in the secondary literature ((DUMMETT, 1991), (HALE, 2000), (ROEPER, 2020), (BOCCUNI; PANZA, 2021)) on the fact that he would have used one or another formulation of Euclid's definition of identity of ratios as an abstraction principle on quantitative domains. The strategy is the following: it is possible to define, in purely logical terms, a relation of proportionality between ordered pairs of elements of a quantitative domain (i.e. Frege's permutations or Wolff's automorphic translations). What interests us is that proportionality is an equivalence relation on pairs of permutations belonging to quantitative domains. Let $\langle f_1, f_2\rangle$ be a pair of permutations belonging to a quantitative domain **Q** and $\langle g_1, g_2\rangle$ be a pair of permutations belonging to a quantitative domain **S**; the following abstraction principle may be introduced:

$$\langle f_1, f_2\rangle \approx \langle g_1, g_2\rangle \quad \Leftrightarrow \quad \text{ratio}(f_1, f_2) = \text{ratio}(g_1, g_2) \qquad (\textbf{APRN})$$

Where '$\approx$' is the relation of proportionality, and 'ratio' is the abstraction operator that given two permutations of a quantitative domain returns their ratio, i.e. a real number. An interesting way of defining proportionality is proposed in (ROEPER, 2020). Roeper defines proportionality between pairs of elements of two quantitative domains (which need not be distinct) as correlation through an isomorphism between structures defined by the fundamental operators of quantitative domains. To formulate this definition, consider the structures $\mathcal{Q}$ and $\mathcal{S}$ respectively associated with the quantitative domains **Q** and **S** and defined as:

$$\mathcal{Q} = \langle \mathbf{Q}, \circ, (\cdot)^{-1}\rangle$$
$$\mathcal{S} = \langle \mathbf{S}, \circ, (\cdot)^{-1}\rangle$$

where '$(\cdot)^{-1}$' is the inverse operator for permutations. The relation of proportionality is defined as follows:

É: Revista Ética e Filosofia Política

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

$$\langle f_1, f_2\rangle \approx \langle g_1, g_2\rangle \quad \underset{def}{\Leftrightarrow} \quad \exists\phi \in \mathbf{S}^{\mathbf{Q}}\,[\mathcal{Q} \overset{\phi}{\cong} \mathcal{S} \;\&\; \phi(f_1) = g_1 \;\&\; \phi(f_2) = g_2] \tag{6}$$

where the expression '$\mathcal{Q} \overset{\phi}{\cong} \mathcal{S}$' denotes the fact that ϕ is isomorphism between $\mathcal{Q}$ and $\mathcal{S}$. It is easy to see that '$\approx$' defined as in (5) is an equivalence relations on pairs of permutations belonging to quantitative domains, thus the abstraction principle (**APRN**) is – at least from a formal perspective – well defined.

Frege's strategy for defining real numbers – as well as its proposed explication – should now be clear. After having noticed that there is no common structure to all physical quantities, Frege introduced the concept of quantitative domain, without discussing the particular nature of the permutations which are elements of quantitative domains. I have proposed to interpret quantitative domains as sets of automorphic translations on a quantitative structure. This move has the advantage of providing a common structure to all continuous quantities. After having demonstrated the fundamental properties of quantitative domains, Frege's strategy may be continued – beyond the point he stopped – by introducing the four-place relation of proportionality, which is an equivalence relation on pairs of permutations of possibly distinct quantitative domains. The relation of proportionality has the interesting feature of being "topic neutral", i.e. it may be defined in higher-order logic (BOCCUNI; PANZA, 2021) and does not require its relata to be autmorphisms on the same quantitative structure. The next step is to introduce the abstraction principle for ratios of permutations (**APRN**), which completes Frege's foundationalist journey through quantities.

(**APRN**) may be used as an important theoretical tool to characterize the logico-metaphysical relations between real numbers and physical quantities; to do this, we need a metaphysical interpretation of abstraction principles, which may be extrapolated from Frege's remarks only to a very limited extent. For this reason we need to move from Frege's texts to the more recent literature on abstractionism.

4 The abstractionist solution to the Pythagorean problem

4.1 Abstraction principles as criteria of identity

The present proposal is based on the idea of using the abstraction principle of real numbers to characterize the relations between real numbers and continuous physical quantities. In particular, these relations are explicated in such a way to formulate a reply to the Pythagoren problem, i.e. to formulate an explanation of the difference between the mathematical continuous structure of reals and the structure of physical continuous quantities. To sketch this strategy it is worth developing an interesting remark on the Pythagoren problem presented at the end of section 2.2.

There seems to be a simple way to say that the structure of real numbers has a quite different nature from the structure of a physical continuous quantity. The mark of physical quantity is a certain structural indeterminacy: it is impossible to refer to a point in a quantitative space using definite descriptions framed in terms of the structural properties of these points. This because the elements of a quantitative structure are identifiable only up to automorphism. On the other hand, if we consider real numbers with all of their structural features (i.e. if we consider the entire field of real numbers) we are considering a completely determined structure and we are able – at least in principles – to identify and refer to every single real number. This because the field of reals has no non-trivial automorphism. As a consequence, we are not able to uniquely assign to every point in a quantitative space a real numbers, yet there will always be a plurality (infinity) of possible numerical attributions.

However, real numbers also form a quantitative structure. Consider the structure $\mathcal{R}$ formed by positive real numbers with addition and order; such a structure admits infinite non-trivial automorphisms: every "scaling function", i.e. every function of the form $f_\alpha(x) = \alpha \cdot x$ with α positive real number distinct from 1, is an automorphic translation on $\mathcal{R}$. Moreover, the set of all translations f_α for every positive real $\alpha \neq 1$, form an ordered Archimedean group. Thus $\mathcal{R}$ is itself a quantity, though a non-physical one. Yet how could we trace a distinction between $\mathcal{R}$ and a alleged continuous physical quantity, e.g. mass

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

or length? According to Wolff's theory physical quantities are physical spaces of points having no haeccetistic identities, while real numbers are not of this sort. If we consider the identity conditions which defines the structure of real numbers, we will see that they are completely identifiable and that $\mathcal{R}$ has a quantitative structure just because we are not considering all features of reals (e.g. we are not considering their behavior with respect to multiplication, division, and substraction). In other words, we are willing to assert that *it is in the "nature" of real numbers (i.e. in their identity conditions) to constitute a fully determinate structure; on the other hand, it is in the "nature" of points of quantitative spaces to constitute partially indeterminate structures, i.e. to be identified only up to automorphism*.

This view is clearly incomplete and unclear, for it must be clarified what are the identity conditions of real numbers and why do they entail a complete structural determinacy. My suggestion is that the fact that real numbers are obtained by abstraction on pairs of automorphic translations in a quantitative space may help to perform two important theoretical tasks: 1) Clarify the identity conditions of real numbers and their ontological dependence on quantitative spaces; 2) Explain the determinacy of mathematical and abstract structures, i.e. the absence of non-trivial automorphisms.

An abstraction principle offers a criterion of identity for the entities it introduces via the abstraction operator. Natural numbers are identical whenever they are associated to sets that can be put in one-to-one correspondence; directions are identical whenever they are associated to parallel lines; real numbers are identical whenever they are associated with proportional pairs of automorphic translations of a quantitative structure (or, in Fregean terms, whenever they are associated to pairs of proportional permutations of quantitative domains). Wright (1983) maintains that the existence of a criterion of identity is necessary and sufficient to sharply define the (sortal) concept under which the objects introduced via abstraction principles fall. According to Wright, natural numbers are the sort of entities which are identified and distinguished in terms of facts of equinumerousity between sets (or concepts); directions are the sort of entities identified and distinguished in terms of facts of parallelism between straight lines; real numbers are the sort of entities which are identified and distinguished in terms of facts of proportionality between automorphic translations of quantitative structures. According to Wright, this fact ensures the reference of abstract terms and a full blooded notion of existence to be ascribed to the referents of

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

terms constructed with abstraction operators. Linnebo (2018), on the other hand, defends a weaker view, according to which the left side of an abstraction principle is only sufficient to ensure the identity of the entities introduced via abstraction and that they may exist in a more lightweight fashion, i.e. a relatively thin objects, objects whose existence requires from world nothing more than what is required by the entities referred to in the left side of an abstraction principle. According to this view, the existence of natural numbers does not require from the world more than what is required by the existence of sets; the same applies to directions and real numbers. Such a view seems more palatable for it does not encounter some critical difficulties of Wright's account which are related to the attribution of a strong conception of objecthood to mathematical objects.

What interests us is that real numbers are the sort of entities which are identified and distinguished by means of facts of proportionality involving automorphic translations of quantitative structures and that these facts are sufficient to ensure reference to real numbers and their existence, even though in a thin sense. In other words, the existence of real numbers demands from the world the existence of quantitative structures, i.e. of structures displaying a certain amount of indeterminacy (i.e. existence of non-trivial automorphisms) and a certain amount of determinacy (i.e. fintely point unique) without which the abstraction principle (**APRN**) would not be an effective criterion of identity for reals. Therefore, by appealing to the fact that the abstraction principle (**APRN**) is the criterion of identity of real numbers we may draw the first desired conclusion: *it is in the "nature" of real numbers (i.e. it is expressed by their criterion of identity) to be thin objects relatively to quantitative structures, i.e. to require the existence of these structures as a sufficient condition for their existence. As a consequence, the structure of real numbers – even when regarded as a quantitative structure – cannot be a quantitative structure on a pair with physical quantitative spaces, for the existence of the latter has explanatory and ontological priority over the existence of the former.* It is quite significant for the understanding of this conclusion to clarify the import of Linnebo's interpretation of abstraction. Without appealing to Linnebo's version of abstractionism, we were not in position to argue that real numbers are a different sort of entities from points in quantitative spaces; this because we would not have at our disposal a criterion of identity for real numbers. To say that real numbers are Dedekind's cuts would not be enough: for we would need an explanation of the fact that Dedekind's cuts are not just ways of set theoretically represent real numbers, yet

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

they *coincide* with real numbers. Abstraction principles seem to overcome this difficulty: there is nothing about the "nature" or identity of the entities appearing in the right side of an abstraction principle over and above the facts expressed by the left side.

The first step of the exposition of my proposal is now complete. We know that real numbers and elements of physical quantitative spaces are distinct sort of entities. However, to provide a complete understanding of this position I must clarify under which respect real numbers may be distinguished from elements of physical quantitative structures. According to Wolff, this distinction should be traced looking at the structures these entities form: real numbers form a completely determined structure whereas points of quantitative spaces form incomplete structures with an infinity of non-trivial automorphisms. Nevertheless, this fact should stem from the criterion of identity of real numbers. Take the case of natural numbers: one may argue that no natural number is identical to Julius Caesar, for persons are not numbers. Yet this reply is not acceptable unless – as Wright attempts to do – we prove that it is a consequence of the chosen criterion of identity for natural numbers that numbers are not persons. Thus the next step will consist in showing that the structural determinacy of real numbers stems from the fact that their criterion of identity is the abstraction principle (**APRN**).

4.2 Abstractionism and structural determinacy

In this section I will argue in favor of the second main claim of my proposal: under certain conditions – which will be stated – abstraction principles may "transform" a partially indeterminate initial structure into a fully determinate structure of thin objects. The basic idea of this claim is simple: partially indeterminate structures are structures which admit non-trivial automorphisms; non-trivial automorphism may be interpreted as symmetries within the initial structure; thus, if we are able to define an equivalence relation whose equivalence classes are classes of structural symmetry, then the abstracts we may introduce by means of such an equivalence relation form a structure with no symmetries. The claim may be presented in an intuitive way by a relatively simple example involving a classical case of abstraction: Hume's Principle. Consider a countably infinite domain D of ur-elements. Suppose that we are to arrange of all subsets of D (i.e. all sets of our ur-elements) into a series of increasing cardinality. This series will start with the empty set and will be

É: Revista Ética e Filosofia Política

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

defined by the relation of "being more numerous than" (and also by the existence of "zero" and succession). Thus we will construe a series starting with the empty set, followed by all singletons, all dupletons, and so on. Clearly, our relation of "being more numerous than" is a partial order, for equinumerous sets will occupy the same position in the series. Moreover, it is easy to see that our structure of cardinalities presents several symmetries, i.e. non-trivial automorphisms: every substitution of a set A with a set B equinumerous to A will preserve all structural facts about A. In other words, equinumerous sets are structurally indistinguishable (though they may be distinguishable in virtue of extra-structural facts). Our structure presents a certain degree of indeterminacy, for every position in our series does not uniquely pick out a set: every choice of a representative set among all sets of the same cardinality is a way of defining a particular position in our structure. We may use an abstraction principle to introduce a new structure –connected to our initial structure of the subsets of D by a homomorphism – which does not display the structural indeterminacy induced by the symmetries of the initial structure. We may say that two sets have the same cardinal number iff there is a one-to-one correspondence between them (i.e. Hume's Principle). Consider now the structure $\mathcal{N}$ of all finite cardinals (and one infinite cardinal). All symmetric elements of the initial structure have been "collapsed" into a unique element of $\mathcal{N}$; as a consequence, every automorphism of the initial structure has been converted by the cardinality operator into a trivial automorphism. Clearly, this is not a proof of the fact that abstraction principles define structures with no non-trivial automorphisms; yet the example seems to be sufficient to offer a preliminary grasp of the fundamental idea, i.e. *If we are able to define an equivalence relation that captures all symmetries of the initial structure, then the structure of abstracts defined using such an equivalence relation will not display these symmetries.*

It is now time to express this idea in rigorous terms, by showing some interesting model theoretic properties of abstraction principles. Firstly, we need a definition which captures the notion of 'structure of abstracts' defined by an abstraction principle on the initial structure. Consider the generic form of a structure:

$$\mathcal{A} = \langle A; R_0, ..., R_n \rangle$$

where $R_i \subseteq A^{k_i}$, for every $i = 0, ..., n$. Consider now an equivalence relation

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

$\sim \subseteq A^2$, such that $\sim$ is a congruence for all relations of $\mathcal{A}$, and a structure:

$$\mathcal{B} = \langle B; S_0, ..., S_n \rangle$$

distinct from $\mathcal{A}$.

Definition 4.1 (Abstracted structure)**.** The structure $\mathcal{B}$ is **the structure abstracted from** $\mathcal{A}$ iff there is a function $h : A \to B$ such that:

(i) $Im(h) = B$;

(ii) For all $x, y \in A$, $x \sim y \Leftrightarrow h(x) = h(y)$;

(iii) For all $i = 0, ..., n$, for all $x_1, ..., x_{k_i} \in A$,

$$R_i(x_1, ..., x_{k_i}) \Leftrightarrow S_i(h(x_1), ..., h(x_{k_i}))$$

* * *

Condition (i) says that the abstracted structure must not include objects which are not introduced by means of the abstraction operator; condition (ii) is the abstraction principle at issue; condition (iii) represents the homomorphism between the initial structure $\mathcal{A}$ and the abstracted structure $\mathcal{B}$.

The next step is to introduce the notion of structural equivalence, i.e. the equivalence relation which captures the idea of structural symmetry. To do this we need the preliminary notion of structural congruence:

Definition 4.2 (Structural congruence)**.** Let $\mathcal{A} = \langle A; R_0, ..., R_n \rangle$ be a structure. The binary relation $\Delta \subseteq A^2$ is a **structural congruence** for $\mathcal{A}$ iff for all $i = 0, ..., n$, for all $x_1, y_1, ..., x_{k_i}, y_{k_i} \in A$,

$$\Delta(x_1, y_1) \,\&\, ... \,\&\, \Delta(x_{k_i}, y_{k_i}) \Rightarrow (R_i(x_1, ..., x_{k_i}) \Leftrightarrow R_i(y_1, ..., y_{k_i}))$$

* * *

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

To say that two elements of the structure $\mathcal{A}$ stand in a relation which is a structural congruence is to say that they are connected through an automorphism. The notion of structural congruence allows us to define a more important notion:

Definition 4.3 (Structural Equivalence)**.** Let $\mathcal{A} = \langle A; R_0, ..., R_n \rangle$ be a structure. The binary relation $\sim \subseteq A^2$ is the **structural equivalence** for $\mathcal{A}$ iff for every $x, y \in A$, $x \sim y$ iff there is a structural congruence Δ such that $\Delta(x, y)$ or $\Delta(y, x)$

* * *

We may use Definition 4.2 to highlight two extreme cases in which structural equivalence is uninteresting, or, in a certain sense, trivial. The first case is when a structure has no symmetries, i.e. only trivial automorphisms; in this case, structural equivalence is the identity relation. The opposite case, is when a structure is *homogeneous*, i.e. when every two elements of the structure (distinct or identical) are connected by some automorphism: in this case, every two elements are structurally equivalent; symmetries are – so to say – "ubiquitous".

The next step should already be clear: I will consider abstraction principles defined using a structural equivalence as equivalence relation and show that the abstracted structure has only trivial automorphisms. This strategy is interesting for two main reasons: 1) If the main criterion of identity for real numbers is based on an abstraction principle using the relation of proportionality and if proportionality is the structural equivalence for the structure of pairs of automorphic translations, then the proposed criterion of identity will explain the main distinguishing feature between mathematical and physical quantities: structural determinacy; 2) Abstraction principles applied to the field of real numbers will not define a new mathematical structure, i.e. structural equivalence for real numbers is the identity relation; on the other hand, no abstraction principle may be interestingly defined on the basic structure of many physical quantities – which are hommogeneous – and thus structural equivalence relates any two elements of the structure. Therefore, our definitions help us to further explicate the strategy of defining real numbers by abstraction of quantities: we need a structure which has certain symmetries – it cannot be a fully determined structure – and yet such a structure must not be homogeneous, i.e. symmetries cannot be ubiquitous. It is for this reason that the abstraction principle for reals cannot be directly applied to

Revista
Ética e
Filosofia Política

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

the universally symmetric structure of automorphic translations – which it may be proven to be homogeneous; and it is precisely for the same reason that we need to introduce a structure having as elements pairs of automorphic translations. In other words, the relation of proportionality between pairs of translations captures a symmetry which is neither trivial – is not an identity – nor ubiquitous – i.e. not all pairs are proportional.

We are now in position to prove the following fundamental result:

Theorem 4.1 (Structural determination by abstraction)**.** *Let $\mathcal{A} = \langle A; R_0, ..., R_n \rangle$ be a structure whose structural equivalence is $\sim$; if $\mathcal{B} = \langle B; S_0, ..., S_n \rangle$ is the structure abstracted from $\mathcal{A}$ with an abstraction operator 'h' and $\sim$ as equivalence relation, then $\mathcal{B}$ has no non-trivial automorphism.*

Proof. We will prove that every automorphism on $\mathcal{B}$ is trivial. Suppose that $f : B \overset{1-1}{\to} B$ is an automorphism on $\mathcal{B}$. Therefore, for every $i = 0, ..., n$, for every $z_1, ..., z_{k_i} \in B$:

$$S_i(z_1, ..., z_{k_i}) \Leftrightarrow S_i(f(z_1), ..., f(z_{k_i}))$$

By clause (i) of Definition 4.1, every element of B is the image of some elements of A through the abstraction operator 'h'. Hence, for every $i = 0, ..., n$, for every $x_1, ..., x_{k_i} \in A$:

$$S_i(h(x_1), ..., h(x_{k_i})) \Leftrightarrow S_i(f \circ h(x_1), ..., f \circ h(x_{k_i}))$$

Define now the following binary relation on A: for all $x, y \in A$

$$\Gamma_f(x, y) \underset{def}{\Leftrightarrow} h(y) = f \circ h(x)$$

By operating the respective substitutions, for every $i = 0, ..., k$ and for all $x_1, y_1, ..., x_{k_i}, y_{k_i} \in A$,

$$\Gamma_f(x_1, y_1) \,\&\, ... \,\&\, \Gamma_f(x_{k_i}, y_{k_i}) \Rightarrow S_i(h(x_1), ..., h(x_{k_i})) \Leftrightarrow S_i(h(y_1), ..., h(y_{k_i}))$$

By clause (iii) of Definition 4.1 and substitution of equivalents, it follows that for every $i = 0, ..., k$ and for all $x_1, y_1, ..., x_{k_i}, y_{k_i} \in A$,

$$\Gamma_f(x_1, y_1) \,\&\, ... \,\&\, \Gamma_f(x_{k_i}, y_{k_i}) \Rightarrow R_i(x_1, ..., x_{k_i}) \Leftrightarrow R_i(y_1, ..., y_{k_i})$$

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

i.e. Γ_f is a structural congruence for $\mathcal{A}$. By Definition 4.2 and clause (ii) of Definition 4.1, for all $x, y \in A$,

$$\Gamma_f(x, y) \Rightarrow x \sim y \Leftrightarrow h(x) = h(y)$$

By definition of Γ_f: for all $x, y \in A$

$$h(y) = f \circ h(x) \Rightarrow h(x) = h(y)$$

Considering that all elements of B are image of some elements of A, for all $z_1, z_2 \in B$:

$$z_2 = f(z_1) \Rightarrow z_1 = z_2$$

i.e. f is the trivial automorphism.

□

The technical details regarding how a structure of pairs of translations may be defined in such a way that proportionality is a structural equivalence are provided in the Appendix. What interests us here is the solution of Pythagorean problem. According to Theorem 4.1, abstraction principles defined in terms of the relation of structural equivalence may be conceived as an operation of structural determination: to every equivalence class of structural symmetry – to every class of proportional pairs of translation – a unique abstract object is associated (in our case, a real number). Thus, *quantitative structures are not mathematical structures insofar as they present certain symmetries that could not be preserved through abstraction; on the other hand – by using abstraction principles as criteria of identity – we are in position to say that it is in the nature of real numbers both to be thin and shallow objects depending on more fundamental facts of proportionality between quantitative translations and that they form a completely determined structure, i.e. a structure with only trivial automorphisms*.

There is a last remark which is worth mentioning. The proposal invert the traditional relation between physical quantities and numbers. The traditional approach in representational theory of measurement is inclined to define physical quantities as physical structures which are representable by means of mathematical structures. Thus real numbers are presupposed to understand what a physical quantity is. According to the present proposal, the perspective is inverted: real numbers are defined in terms of proportionality of pairs of translations on quantitative structures, therefore, physical quantities are logically and metaphysically more fundamental than the mathematical structures we use to represent them.

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Appendix

In this section I will sketch how a structure of pairs of automorphic translations having non-trivial automorphisms and for which proportionality is a structural equivalence may be construed.

Definition 4.4 (*Automorphic product*). Let $\mathbf{Q}$ be a quantitative structure, and $\mathcal{T}(\mathbf{Q})$ be the group of automorphic translations on $\mathbf{Q}$. For all $f_1, f_2, g_1, g_2, h_1, h_2 \in \mathcal{T}(\mathbf{Q})$ (with f_1, f_2, g_1, g_2 distinct from the identity function), the **automorphic product** $\langle f_1, f_2\rangle \cdot \langle g_1, g_2\rangle$ is defined as follows:

Número XXVI - Volume I - julho de 2023
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ISSN: 1414-3917
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$$\langle f_1, f_2\rangle \cdot \langle g_1, g_2\rangle \approx \langle h_1, h_2\rangle$$

iff exist two automorphisms ϕ, ψ on $\mathcal{T}(\mathbf{Q})$ such that for all $u \in \mathcal{T}(\mathbf{Q})$:

$$\phi(u) = f_1 \text{ and } \psi(u) = f_2 \quad \Rightarrow \quad \langle \phi(g_1), \psi(g_2)\rangle \approx \langle h_1, h_2\rangle$$

and if f_1 or g_1 is the identical function e, then $h_1 = e$

* * *

Theorem 4.2 (*Properties of the automorphic product*)**.** *If $\mathcal{T}(\boldsymbol{Q})$ is homogeneous and one point unique, then for all $f_1, f_2, g_1, g_2, h_1, h_2 \in \mathcal{T}(\boldsymbol{Q})$ distinct from the identity function:*

(i) $\langle f_1, f_2\rangle \cdot (\langle g_1, g_2\rangle \cdot \langle h_1, h_2\rangle) \approx (\langle f_1, f_2\rangle \cdot \langle g_1, g_2\rangle) \cdot \langle h_1, h_2\rangle$ *(Associativity);*

(ii) $\langle g_1, g_1\rangle \cdot \langle f_1, f_2\rangle \approx \langle f_1, f_2\rangle$ *(Neutral elements);*

(iii) If $\mathcal{T}(\boldsymbol{Q})$ is a commutative group, then $\langle f_1, f_2\rangle \cdot \langle f_2, f_1\rangle \approx \langle g_1, g_1\rangle$ (Inverse elements);

(iv) If $\mathcal{T}(\boldsymbol{Q})$ is a commutative group, then $\langle f_1, f_2\rangle \cdot \langle g_1, g_2\rangle \approx \langle g_1, g_2\rangle \cdot \langle f_1, f_2\rangle$ (Commutativity);

Proof. **Proof of item (i).** Suppose that

$$\langle f_1, f_2\rangle \cdot (\langle g_1, g_2\rangle \cdot \langle h_1, h_2\rangle) \approx \langle l_1, l_2\rangle$$

by applying the definition of automorphic product, exists $u \in \mathcal{T}(\mathbf{Q})$ and ϕ, ψ automorphisms of $\mathcal{T}(\mathbf{Q})$ such that:

$$\langle g_1, g_2\rangle \cdot \langle h_1, h_2\rangle = \langle \phi(u), \psi(u)\rangle \cdot \langle h_1, h_2\rangle \approx \langle \phi(h_1), \psi(h_2)\rangle$$

Hence, by the fact that the automorphic product is invariant by substitutions of proportional pairs:

$$\langle f_1, f_2\rangle \cdot \langle \phi(h_1), \psi(h_2)\rangle \approx \langle l_1, l_2\rangle$$

Número XXVI - Volume I - julho de 2023
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ISSN: 1414-3917
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Exists $v \in \mathcal{T}(\mathbf{Q})$ and two automorphisms η, θ such that:

$$\langle f_1, f_2\rangle = \langle \eta(v), \theta(v)\rangle \quad \text{and} \quad \langle l_1, l_2\rangle \approx \langle \eta \circ \phi(h_1), \theta \circ \psi(h_2)\rangle$$

Consider now the expression:

$$(\langle f_1, f_2\rangle \cdot \langle g_1, g_2\rangle) \cdot \langle h_1, h_2\rangle$$

By substitutions of identicals and proportionals, we obtain the proportional expression:

$$(\langle \eta(v), \theta(v)\rangle \cdot \langle \phi(u), \psi(u)\rangle) \cdot \langle h_1, h_2\rangle$$

which is in turn proportional to:

$$(\langle \eta \circ \phi(u), \theta \circ \psi(u)\rangle \cdot \langle h_1, h_2\rangle \approx \langle \eta \circ \phi(h_1), \theta \circ \psi(h_2)\rangle \approx \langle l_1, l_2\rangle$$

q.e.d..

Proof of item (ii). Consider the expression:

$$\langle g_1, g_1\rangle \cdot \langle f_1, f_2\rangle$$

By definition of automorphic product:

$$\langle g_1, g_1\rangle = \langle \xi(u), \eta(u)\rangle \quad \text{and} \quad \langle g_1, g_1\rangle \cdot \langle f_1, f_2\rangle \approx \langle \xi(f_1), \eta(f_1)\rangle$$

For some $u \in \mathcal{T}(\mathbf{Q})$ and automorphisms ξ, η. By the fact that $\xi(u) = \eta(u)$ and that $\mathcal{T}(\mathbf{Q})$ is one point unique, it follows that $\xi = \eta$. Therefore:

$$\langle g_1, g_1\rangle \cdot \langle f_1, f_2\rangle \approx \langle \xi(f_1), \xi(f_2)\rangle \approx \langle f_1, f_2\rangle$$

q.e.d. **Proof of item (iii).** Consider the expression:

$$\langle f_1, f_2\rangle \cdot \langle f_1, f_2\rangle$$

By definition of automorphic product:

$$\langle f_1, f_2\rangle = \langle \xi(u), \eta(u)\rangle \quad \text{and} \quad \langle f_1, f_2\rangle \cdot \langle f_2, f_1\rangle \approx \langle \xi(f_2), \eta(f_1)\rangle$$

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
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It follows that:

$$\langle \xi(f_2), \eta(f_1) \rangle = \langle \xi \circ \eta(u), \eta \circ \xi(u) \rangle$$

By commutativity of '$\circ$' in $\mathcal{T}(\mathbf{Q})$:

$$\langle \xi \circ \eta(u), \eta \circ \xi(u) \rangle = \langle \eta \circ \xi(u), \eta \circ \xi(u) \rangle = \langle \eta(f_1), \eta(f_1) \rangle \approx \langle f_1, f_1 \rangle$$

Proof of item (iv). Assume that for some $u, v \in \mathcal{T}(\mathbf{Q})$ and automorphisms xi, η, λ, μ:

$$f_1 = \xi(u) \ , \ f_2 = \eta(u) \ , \ g_1 = \lambda(v) \ , \ g_2 = \mu(v)$$

Hence:

$$\langle f_1, f_2 \rangle \cdot \langle g_1, g_2 \rangle = \langle \xi(u), \eta(u) \rangle \cdot \langle \lambda(v), \mu(v) \rangle \approx \langle \xi \circ \lambda(u), \eta \circ \mu(v) \rangle = \langle \lambda \circ \xi(v), \mu \circ \eta(v) \rangle$$

$$\langle g_1, g_2 \rangle \cdot \langle f_1, f_2 \rangle = \langle \lambda(v), \mu(v) \rangle \cdot \langle \xi(u), \eta(u) \rangle \approx \langle \lambda \circ \xi(u), \mu \circ \eta(u) \rangle$$

By homogeneity of $\mathcal{T}(\mathbf{Q})$, exists an automorphism ϕ such that $u = \phi(v)$; as a consequence:

$$\langle \lambda \circ \xi(u), \mu \circ \eta(u) \rangle = \langle \lambda \circ \eta \circ \phi(v), \mu \circ \eta \circ \phi(v) \rangle = \langle \phi \circ \lambda \circ \xi(v), \phi \circ \mu \circ \eta(v) \rangle \approx \langle \lambda \circ \xi(v), \mu \circ \eta(v) \rangle$$

Therefore,

$$\langle f_1, f_2 \rangle \cdot \langle g_1, g_2 \rangle \approx \langle g_1, g_2 \rangle \cdot \langle f_1, f_2 \rangle$$

$\square$

Definition 4.5 (*Binary sum*)**.** Let $\mathbf{Q}$ be a quantitative structure, and $\mathcal{T}(\mathbf{Q})$ be the group of automorphic translations on $\mathbf{Q}$. For all $f, u, g, h \in \mathcal{T}(\mathbf{Q})$ distinct from the identity function, the **binary sum** $\langle f, u \rangle \oplus \langle g, u \rangle$ is defined as follows:

$$\langle f, u \rangle \oplus \langle g, u \rangle \approx \langle f \circ g, u \rangle$$

* * *

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For the properties of binary sum (i.e. associativity, inverse elements, and neutral elements) see (ROEPER, 2020).

It is now possible to define the structure of pairs of automorphic translations; for the sake of simplicity we will consider only pairs of translations over the same quantitative structure (the extension to the case of different quantitative structures presents no conceptual difficulty):

Definition 4.6 (Fraction of magnitudes)**.** The structure:

$$\mathcal{F}(\mathbf{Q}) = \langle [\mathcal{T}(\mathbf{Q})]^2, \approx, \cdot, \oplus \rangle$$

is **the structure of fractions of magnitudes** of **Q**.

* * *

Theorem 4.3 (Representation theorem)**.** *Let $\mathcal{F}(\boldsymbol{Q})$ be a structure of fractions of magnitudes. For every automorphism Φ of $\mathcal{F}(\boldsymbol{Q})$, there is an automorphism ξ of $\mathcal{T}(\boldsymbol{Q})$, such that for every $x, y \in \mathcal{T}(\boldsymbol{Q})$,*

$$\Phi(x, y) = \langle \xi(x), \xi(y) \rangle$$

Proof. The following notations will be useful:

$$\Phi(x, y) = \langle \Phi_1(x, y), \Phi_2(x, y) \rangle$$

$$\vec{\alpha}(x) = \langle \alpha(x), \alpha(x) \rangle$$

It is easy to verify – from the definition of binary sum – that the following holds for all $x, y, z \in \mathcal{T}(\mathbf{Q})$ and some automorphisms α, β of $\mathcal{T}(\mathbf{Q})$:

$$\Phi(x \circ y, z) \approx \vec{\alpha}(\Phi(x, z)) \oplus \vec{\beta}(\Phi(y, z)) \qquad (*)$$

Let e the identity function; for all $a, u \in \mathcal{T}(\mathbf{Q})$:

$$\langle a, u \rangle \oplus \langle a^{-1}, u \rangle \approx \langle a \circ a^{-1}, u \rangle \approx \langle e, u \rangle$$

Revista
Ética e
Filosofia Política

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
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By definition of automorphism:

$$\Phi(a,u) \oplus \Phi(a^{-1},u\rangle \approx \Phi(a \circ a^{-1},u) \approx \langle e,u\rangle$$

where the property of the neutral element of being a fixed point has been used. By $(*)$:

$$\Phi(a \circ a^{-1},u) \approx \vec{\alpha}(\Phi(a,u)) \oplus \vec{\beta}(\Phi(a^{-1},u)) \approx \langle e,u\rangle \qquad (**)$$

Using the definition of binary sum and the notational convention (and omitting some parentheses):

$$\langle \alpha\Phi_1(a,u), \alpha\Phi_2(a,u)\rangle \oplus \langle \beta\Phi_1(a^{-1},u), \beta\Phi_2(a^{-1},u)\rangle \approx \langle \alpha\Phi_1(a,u) \circ \beta\Phi_1(a^{-1},u), \alpha\Phi_2(a,u)\rangle$$

where α and β are such that $\alpha\Phi_2(au) = \beta\Phi_2(a^{-1},u)$. By $(**)$ and the fact that e is a fixed point of all automorphisms:

$$\alpha\Phi_1(a,u) \circ \beta\Phi_1(a^{-1},u) = e$$

From which it follows that:

$$[\Phi_1(a,u)]^{-1} = \alpha^{-1} \circ \beta\Phi_1(a^{-1},u)$$

In other words, for all $x, y \in \mathcal{T}(\mathbf{Q})$, and for all autmorphism Φ of $\mathcal{F}(\mathbf{Q})$, there is an automorphism ξ of $\mathcal{T}(\mathcal{Q})$, such that:

$$[\Phi_1(x,y)]^{-1} = \xi\Phi_1(x^{-1},y) \quad \text{and} \quad \Phi_2(x,y) = \xi\Phi_2(x^{-1},y) \qquad \textbf{(1)}$$

By similar considerations, it is possible to show that:

$$[\Phi_1(e,x)] = e \quad \text{and} \quad \Phi_2(e,x) = \xi\Phi_2(y,x) \qquad \textbf{(2)}$$

From **(1)** and **(2)** two important facts follow: that the values of Φ_1 and Φ_2 depends respectively only upon the first and the second argument of Φ; and that Φ_1 is an automorphism of $\mathcal{T}(\mathbf{Q})$. Therefore, every automorphism Φ of $\mathcal{F}(\mathbf{Q})$ may be written as:

$$\Phi(x,y) = \langle \phi(x), \Phi_2(y)\rangle$$

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

for some ϕ automorphism of $\mathcal{T}(\mathbf{Q})$.

Consider now the following expression:

$$\langle a, b\rangle \cdot \langle u, u\rangle \approx \langle a, b\rangle$$

valid for all $a, b, u \in \mathcal{T}(\mathbf{Q})$. Being Φ and automorphism of $\mathcal{F}(\mathbf{Q})$:

$$\Phi(a, b) \cdot \Phi(u, u) \approx \Phi(a, b)$$

By developing the previous expression:

$$\langle \phi(a), \Phi_2(b)\rangle \cdot \langle \phi(u), \Phi_2(u)\rangle \approx \langle \phi(a), \Phi_2(b)\rangle$$

By definition of automorphic product, exist ξ, η automorphisms of $\mathcal{T}(\mathbf{Q})$ and $w \in \mathcal{T}(\mathbf{Q})$ such that:

$$\xi(w) = \phi(a)\ ;\ \eta(w) = \Phi_2(b)\ ;\ \langle \xi \circ \phi(u), \eta \circ \Phi_2(u)\rangle \approx \langle \phi(a), \Phi_2(b)\rangle$$

By definition of proportionality, there is an automorphism ψ of $\mathcal{T}(\mathbf{Q})$ such that:

$$\xi \circ \phi(u) = \psi \circ \phi(a) = \psi \circ \xi(w)\ ;\ \eta \circ \Phi_2(u) = \psi \circ \Phi_2(b) = \psi \circ \eta(w)$$

Explicitating w and by commutativity of $\circ$:

$$^{-1} \circ \phi(u) = \psi^{-1} \circ \Phi_2(u)$$

Given that the previous expression is valid for all $u \in \mathcal{T}(\mathbf{Q})$, if follows that $\phi = \Phi_2$. As a consequence, for every automorphism Φ of $\mathcal{F}(\mathbf{Q})$, there is an automorphism ϕ of $\mathcal{T}(\mathbf{Q})$ such that for all $x, y \in \mathcal{T}(\mathbf{Q})$:

$$\Phi(x, y) = \langle \phi(x), \phi(y)\rangle$$

□

Theorem 4.4 (Proportionality as structural equivalence.)**.** *The relation of proportionality is a structural equivalence for the structure of fraction of magnitudes $\mathcal{F}(\boldsymbol{Q})$.*

Proof. The relation '$\equiv$' of structural equivalence for $\mathcal{F}(\mathbf{Q})$ may be defined as follows:

Revista
Ética e
Filosofia Política

Número XXVI - Volume I - julho de 2023
https://periodicos.ufjf.br/index.php/eticaefilosofia
ISSN: 1414-3917
e-ISSN: 2448-2137

$\langle x, y\rangle \equiv \langle x'y'\rangle$ iff exists an automorphism Φ of $\mathcal{F}(\mathbf{Q})$ such that $\Phi(x,y) = \langle x'y'\rangle$ or $\Phi(x'y') = \langle x, y\rangle$

By the previous theorem, the definition is equivalent to:

$$\langle x, y\rangle \equiv \langle x'y'\rangle$$

iff exists an automorphism ϕ of $\mathcal{T}(\mathbf{Q})$ such that

$$\phi(x) = x' \text{ and } \phi(y) = y' \text{ or } \phi(x') = x \text{ and } \phi(y') = y$$

which is the definition of the relation of proportionality.

$\square$