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NOT COPED WITH BY A MACHINE: ON FREGE'S CONCEPTION OF LOGIC AS SCIENCE

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Abstract: Following logicism, Frege famously held that logic is a science on its own. Particularly, he held the informativity thesis, *viz.,* that logic is a science because it is deductively informative. This paper aims to understand Frege's informativity thesis and its connection with the conception of logic as science. For such, it focuses on some features of Frege's philosophy that are key for understanding this connection, particularly his conception of analyticity, the role of judgments in inferential reasoning, and the use of decomposition of functions as a means for concept-formation, which Frege names as fruitful definitions.

Key-words: Gottlob Frege; Informativity of Logic; Judgements; Fruitful Definitions.

> *[...] our thinking as a whole can never be coped with by a machine or replaced by purely mechanical activity.*
>
> Gottlob Frege

1 Introduction: the Scandal of Deduction

Logic is regarded as a deductive form of reasoning since, in any logically valid argument, the conclusion is supported solely by its premises: it simply restates the premises in such a way that nothing new is added. If something else is added, then we lose the formal validity of the argument. That's why logical truths are said to be tautological or analytical. On the other hand, it seems clear that we, in fact, gain knowledge when we do logic, i.e. we accept the informativity of the conclusion, even though it suppose to be

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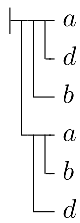
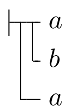
nothing more than what we already know as premises. In other words, we accept that the conclusion was somehow unexpected, or at least that we didn't know it in advance. That is one of the main reasons to do logic: to verify and confirm the truth of some propositions.

To illustrate this, consider the following example. In 1879, Frege published its famous *Begriffsschrift*, with an axiomatic treatment for propositional and predicative logic (FREGE, 1967). One of the axioms, labeled formula (8)[1], is the following:



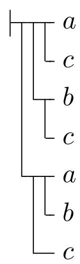
(8)

Later, in 1929, Łukasiewicz (1970) discovered that this axiom was not independent, being provable from Frege's other axioms[2], particularly from:



(1)

and



(4)

This is a simple fact of propositional logic. One need's nothing more than both axioms and the rules from propositional calculus to derive the former axiom as a theorem. Frege's unawareness of this "obvious" conclusion exemplifies a simple fact: we are not aware of the whole set of conclusions that a set of premises logically have. But at the same time, it is expected that we *should* be since the validity of the deduction depends upon the fact that nothing spurious is added to the proof. If we have all the tools, we should know their consequences. But this, as the example above shows, is clearly false.

This is an old problem, recast by Hintikka (1970) as the *scandal of deduction*: the fact that, although being essentially tautological, no satisfactory account has been given for the highly informational capabilities of logical truths. The natural answer, the one provided by Wittgenstein and the Vienna Circle, is that

[1] Avoiding Frege's cumbersome notation, this is $\vdash (d \to (b \to a)) \to (b \to (d \to a))$

[2] In modern notation, these are $\vdash a \to (b \to a)$ and $\vdash (c \to (b \to a)) \to ((c \to b) \to (c \to a))$, respectively.

both logic and mathematics are *per se* tautological. By being so, no information could possibly be carried out through deductions. Any information that we actually gain in theorem-proving is only of psychological relevance. In the Fregean example above, it implies that it was a matter of psychological limitations that prevented him from realizing the fact later proved by Łukasiewicz, as it would demand only a little more effort for him to realize it too. As a consequence, Frege would accept the result with surprise, being obviously unaware of it.

This psychological perspective suggests, as Hintikka puts it, the far-fetched idea that the work of a logician or mathematician resembles some "therapeutic exercises calculated to ease the psychological blocks and mental cramps that initially prevented us from being [...] 'aware of all that we implicitly asserted' already in the premises" (HINTIKKA, 1970, 35-136). It would be as if our lack of awareness were simply a matter of lack of resources to efficiently compute all the logical consequences provided by a given proposition. This psychological account is hardly acceptable. Not only are we not omniscient towards deductions, but it's also equally hard to accept that mathematics and logic, as it seems, are just *apparently* informative.

Nonetheless, this was the dominant view under logical positivism of the Vienna Circle, made clear by their rejection of the Kantian epistemology:

> It is precisely in the rejection of the possibility of synthetic knowledge a priori that the basic thesis of modern empiricism lies. The scientific world-conception knows only empirical statements about things of all kinds, and analytic statements of logic and mathematics. (NEURATH, 1973, p.308)

But despite rejecting the existence of synthetic *a priori* judgments, the Vienna Circle still adopted the Kantian position that analytic judgments are unable to extend knowledge. In fact, their manifesto took as a mistake

> [...] the notion that thinking can either lead to knowledge out of its own resources without using any empirical material, or at least arrive at new contents by an inference from given states of affair. Logical investigation, however, leads to the result that all thought and inference consists 'of nothing but a transition from statements to other statements that contain nothing that was not already in the former (tautological transformation). (NEURATH, 1973, 308)

The influence of the early Wittgenstein of the *Tractatus Logico-Philosophicus* (2002) is clear and can be found in the following passages:

> 4.561 Tautologies and contradictions lack sense.
>
> 6.1 The propositions of logic are tautologies.
>
> 6.11 Therefore the propositions of logic say nothing. (They are the analytic propositions.)
>
> 6.21 A proposition of mathematics does not express a thought.

Thus, given that both the early Wittgenstein and the Vienna Circle equated logic with mathematics, and that both accepted the uninformative character of logic, they held the controversial thesis that mathematics is *a fortiori* uninformative.

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This view on the informativity of logic was followed in the ’50s by Bar-Hillel and Carnap’s (1964) theory of semantic information. The theory follows the Inverse Relationship Principle, in which the amount of information available for a sentence is correlated with its unpredictability: the more improbable it is, the more information it carries. In a modal framework, this means that the more possible worlds in which a given sentence is true, the less informative it is. The semantic content of a sentence s is identified with the worlds, or state descriptions, excluded by s, *i.e.* those worlds in which s is false. Hence, the measure of semantic content of s is the complement of the probability of s. It is evident, in this account, that if s is a logical truth, a tautology, its devoid of content since the complement set of the worlds in which is true is empty following s being true in every world or state description. This means that logical truths have no content and, thus, they don’t convey any information as well. This implies that the scandal of deduction is a problem for Bar-Hillel and Carnap’s theory of semantic information. Now, if s is a contradiction, then s is false at every state description or possible world. In this case, s has the highest amount of semantic content and conveys too much information. This oddity is now regarded as the Bar-Hillel-Carnap Paradox.

These problems have strengthened the debate on the informativity of logic. Kant already defended the idea that logical reasoning is incapable of extending knowledge, as he thought formal logic to be a *cânon* of reason.[3] And as we already pointed out above, the same conclusion was held by the early Wittgenstein and the Vienna Circle, albeit on different grounds, as they rejected Kantian epistemology altogether. Nonetheless, a third and possible option was already known and defended by Gottlob Frege in order to advance logicism, as he sought not to reject Kantian epistemology entirely[4] but only the assumption that formal logic is a *cânon* for reason[5]

Just as Wittgenstein and the Vienna Circle, Frege equally identified mathematics (at least arithmetic and number theory) with logic. But far from accepting the sterile conception of analyticity, Frege defended that mathematics is a contentful science and, therefore, that logic is equally as informative. Therefore, Frege defended that logic is a proper science capable of real extensions of knowledge. The aim of this paper is to make sense of Frege’s conception of logic as science through the thesis of the informativity of logic. Frege’s claim on it, which grants its scientific status, can be understood as a product of some features of his philosophy and the concept-script’s proof system, particularly his conception of analyticity, the fruitfulness of definitions and the role of decompositions in concept formation, and finally, the role of judgments in inferential reasoning. Although the concept-script is distinct from modern logical systems, the conclusion from these three features is a particular, but still interesting, answer to the scandal of deduction, which comes down to Frege’s conception of logic as science.

[3] See, for instance, Kant (1998, B74).
[4] For example, Frege still agreed with Kant that Geometry is synthetic *a priori*.
[5] This rejection was a common assumption made by nineteenth-century Logicism, who took Kant’s conception of the synthetic *a priori* as the main target, as explained by Coffa (1982).

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2 Frege's Concept-Script

Frege's system of logic, the concept-script, needs no introduction, as it is well established as one of the first systematic formalizations of quantificational logic. There were two versions of the concept-script: the first published in 1879 in a small booklet entitled *Begriffschrift* (1967)[6], and the supplemented and altered version in 1893's logicist manifesto *Grundgesetze der Arithmetik* (2013). In modern terms, both systems are second-order logics. Since Frege's claims about the informativity of logic are mostly made before the renowned 1893 version of the concept-script, we can focus on the 1879 version.[7]

The concept-script uses a two-dimensional writing, and the syntax follows Frege's definitions of *strokes*:

$$—\quad ,\top\quad ,\quad \big| \;,\smile\quad ,\quad \equiv\quad ,\quad |\quad ,\quad ||$$

These are called the content, negation, conditional, concavity (for generality), identity, judgment, and double definitional strokes. With these, Frege adds Roman and Gothic letters to express content generality. First, he explicitly abandons the subject-copula-predicate structure of judgments by importing functional symbols from mathematics. Thus, the following letters are added:

1. Lower case Roman letters $a, b, c, ...x, y, z, ...$ for argument variables;
2. Lower case Roman letters $f^n, g^n, h^n, ...$ as n-ary functional variables.
3. Lower case Gothic letters $\mathfrak{a}$,$\mathfrak{b}$,$\mathfrak{c}$... for bounded individual variables;
4. Upper case Gothic letters $\mathfrak{F}^n$, $\mathfrak{G}^n$, $\mathfrak{H}^n$ for bounded functional variables.

Frege uses the content-stroke in order to distinguish those contents that *can* be judged from those who can't: ''Whatever follows the content stroke must always have an assertible content'' (FREGE, 1967, §2). Thus, whatever follows it must be a content of possible judgment, which could be loosely read as those having a truth-value[8]. Although Frege does not mention it, the content-stroke can be used to define the term-formula distinction within the concept-script. Terms can be defined in the following way:

- For every argument variable x, if x has an individual as value, then x is a term;
- For every function variable f which is not a conceptual function, and argument variable x, if x is a term, then $f(x)$ is a term.[9]

[6] All Frege's work quoted are taken directly from canonical English translations.

[7] I do not claim this assessment to be the full Fregean picture for the informativity of and scientific character of logic. In fact, there is much to unveil about the topic coming from the mature Frege, including the important sense-reference distinction. Nonetheless, Frege's claims are tied up with the earlier version of his system, and going beyond it would be too much ground to cover here.

[8] Frege does not speak about truth-values in the 1879 version of the concept-script. Instead, he uses expressions such as ''the proposition of A'' or ''the circumstance that A'' (FREGE, 1967, §2) to read the application of the content-stroke.

[9] Numerals such as '2' are examples of the former, while operations such as '2+2' are examples of the latter.

Given that terms are not assertible, they cannot follow the content-stroke. However, if a function f describes a concept, the application of f to a term x defines atomic formulas in the modern sense:

- If f denotes a concept and x is a term, then $f(x)$ is an atomic formula:

$$— f(x).$$

Besides being a marker for conceptual-contents in Frege's terminology (FREGE, 1967, §3), the content-stroke thus function as a marker for well-formed formulas. In fact, if $— \alpha$ and $— \beta$ are atomic formulas, then

$$— \alpha \qquad \begin{array}{l} \top\!\!— \beta \\ \;\llcorner\, \alpha \end{array} \qquad \top \alpha \qquad —(\alpha \equiv \beta) \qquad —\!\overset{\mathfrak{a}}{\smile}\!— \alpha$$

are all well-formed formulas. These are read, respectively, as the content of α (FREGE, 1967, §1), the material implication between α and β (§5), the negation of α (§7), the identity between the contents of α and β (§8),[10] and, finally, the generality of α (§11).

In the modern model-theoretic practice, we would define truth in terms of models in the usual Tarskian way, provided the adoption of a richer metalanguage. None of these tools were available to Frege.[11] Instead, Frege's logic follows the tradition of adopting judgments.[12] As the tradition often treats judgments as categorical propositions, Frege's formalization introduces a distinction between content and the judgment proper. According to (FREGE, 1967, §3), the content-stroke functions as a nominalization operator[13], separating the content from what he later called the assertoric force. Without such force,

$$— \alpha$$

reads as ''the circumstance of α''.[14] To make a judgment, he introduces the judgment-stroke:

$$\vdash \alpha$$

which expresses the act of judging $— \alpha$ as true or as being the case.

Judging is an inner event: something that happens in the mind of an agent. Classically, a judgment is a form of decision: one decides the truth-value of the content being considered, either affirming it (as being true) or denying it (as being false). In Frege's case, this decision is whether to judge it as true or not

[10] The identity of contents-sign is an interesting exception, as Frege wants the sign to be applicable both to well-formed formulas and to terms, meaning that the content-stroke is not required for the signs flanking the identity-function. It adds a bifurcation to the system, as the sign is defined over expressions of the language, not to the contents being expressed by those signs (FREGE, 1967, §3). We shall not bother to the oddity of this sign, though.

[11] Not only did Frege adopt the view that truth is *sui generis* and indefinable (FREGE, 1984, 353), there is no metatheory in the modern sense in Frege's concept-script. On this, see Goldfarb (2010) and Weiner (2005).

[12] On this tradition, see Sundholm (2008).

[13] See Bell (1979) on this reading.

[14] In the later version of the concept-script, this would be read as ''the truth-value of α'', following the introduction of truth-values as objects.

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to judge at all, as there is no sign for the act of rejection in his system. Having made a judgment, if one wants to communicate it, he uses the judgment-stroke. Frege distinguishes between *judging* a content, and *communicating* the judgment, what he calls the *assertion*: ''to recognize something as true is to make a judgement, and to give expression to this judgement is to make an assertion'' (FREGE, 1979, p.2).[15]

For these reasons, the judgment-stroke is a *sui generis* sign. As Frege says, it is a ''sign of his own kind'' (FREGE, 2013, §26), since it ''does not designate anything; it asserts something'' (FREGE, 1984, p.149), given that it is, in modern terms, a marker for the illocutionary force of an assertion. This means that prefixing a content with the judgment-stroke is to perform a Speech Act.[16] Therefore, in asserting ''$— \alpha$'' with the judgment-stroke, *viz.,* $\vdash \alpha$, one is acknowledging and asserting that $— \alpha$ is true.[17]

The presence of an illocutionary force indicating device in the concept-script makes it an entirely different system of logic if compared to modern model-theoretic systems. Given that proofs in the system are sequences of asserted formulas, all deductions are *ipso facto* inferences[18]. While modern systems use models to formalize logical consequence as a metatheoretical relation holding between two formulas ($\alpha \models \beta$), in the concept-script, one goes from the assertion of one formula to the assertion of another, following the rules of inference. In fact, there is only one rule of inference in the concept-script: *modus ponens*. Accordingly, instead of taking *modus ponens* as the set of all ordered pairs in the form $\langle\langle\alpha, \alpha \to \beta\rangle, \beta\rangle$[19], Frege defines it using the permissive mood: if the premises are asserted, the conclusion shall be asserted. Moreover, assertions are belief attitudes, which by itself makes the concept-script epistemically driven, and the ideal tool for scientific discoveries.

3 Analyticity and the Informativity Thesis

The sheer presence of judgments already makes logic epistemically oriented, given that to assert something is to acknowledge it as true. This epistemic import was applicable to arithmetic, as Frege believed. It meant that we could derive arithmetical truths from logical laws in an analytical but still informative way. Most of Frege's remarks on this point are found in the *Grundlagen der Arithmetic* (FREGE, 1953). He questions, "[...] how do the empty forms of logic come to disgorge so rich a content?" (FREGE, 1953, §16). The fact that arithmetic is a contentful science is an unquestionable premise for him. If it is possible to show that arithmetic is just a developed logic, then

> [...] the prodigious development of arithmetical studies, with their multitudinous applications, will suffice to put an end to the widespread contempt for analytic judgements and to the legend of the sterility of pure logic.(FREGE, 1953, §17).

[15] See also Frege (1984, p.355-6).

[16] See Searle (1979) for a brief assessment of modern Speech Act theory. The idea of reading Frege's judgement stroke from the Speech Act perspective was already hinted at by Bell (1979). It was later restated by Ruffino, San Mauro and Venturi (2021). A more developed version is found in Schmidt (2021).

[17] Later, Frege defines the judgment-stroke by saying ''I understand by a judgement the acknowledgement of the truth of a thought'' (FREGE, 2013, §5).

[18] On the distinction between logical consequence and inferences in the history of logic, see Sundholm (2008).

[19] See, for instance, Mendelson (2015, p.27).

This is Frege's version of the scandal of deduction, given in Kantian terminology: how can analytic judgments be informative?

Analyticity and logicality are tied together in Kant's philosophy. If a given argument is deductively valid, it is said to be analytically valid. A deductive or analytical form of reasoning is often in contrast to synthetic forms of reasoning. In the first *Critique*, Kant defines analytic judgments in terms of containment between the concepts related: "A is B" is analytically true just in case $B \in_R A$[20], and synthetically true otherwise (KANT, 1998, B10). Moreover, Kant stresses that in analytical judgments, we are justified in judging the truth of the connection by non-contradiction: "we must also allow the *principle of contradiction* to count as the universal and completely sufficient *principle of all analytic cognition*" (KANT, 1998, B191).

Analytic judgments, being grounded on logic alone, are not capable of extending knowledge, according to Kant. For that reason, he calls them *judgments of clarification*. Even though formal logic is capable of judging with synthetic judgments, it is incapable of judging beyond them. This is because Kant sees logic as only a formal canon for reasoning: it "abstracts from all contents of the cognition of the understanding and of the difference of its objects, and has to do with nothing but the mere form of thinking" (KANT, 1998, B78).[21] By being so, formal logic does not have proper objects. It is an uninterpreted formal calculus and is not a science for the purpose of extending knowledge. It is not "[...] a universal art of discovery [...] and not an organon of truth - not an algebra, with whose help hidden truths can be discovered" (KANT, 1992, p.534), Kant says in the *Jäsche Logic*.

Thus, Kant has a negative answer to the scandal of deduction: analytic judgments cannot extend knowledge. In his account, it is not possible for a logically valid argument to convey any information that was not already contained in the premises. And given that Kant also accepts that mathematics is a proper science, that is, has an intended interpretation, it follows that mathematics cannot be founded in logic alone. This is the background of Frege's attempt to show that analytic judgments, *contra* Kant, can indeed extend knowledge. Kant had, for him, "[...] underestimated the value of analytic judgements" by "[...] defining them too narrowly"(FREGE, 1953, §88).

For that matter, Frege claims that analyticity does not concern ''[...] the content of the judgement but the justification for making the judgement'' (FREGE, 1953, §3). In other words, analyticity becomes a feature dependent on the inference leading to a judgment, not on its content. As he explains,

> The problem becomes, in fact, that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one. [...] If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the proposition is a synthetic one. (FREGE, 1953, §3)

Thus, a judgment is fregean-analytic if it has a purely logical proof, that is, if it follows from ba-

[20] The binary relation $\in_R$ should be read as a containment between mental representations, in Kant's philosophy. Since mental representations are composed of other mental representations, $B \in_R A$ means simply that B is a subpart of A.

[21] This is what Kant calls the *General Logic*. By contrast, it is the Transcendental Logic that deals with the proper connection between concepts and the objects of experience.

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sic logical laws, definitions, and truth-preserving rules of inferences. Given that proofs are inferences in Frege's concept-script, and since inferences are transitions from judgments to other judgements, the first step towards informativity is provided. Any conclusion, being itself a judgment, does carry information since the act of asserting it as true shows that the agent believes it to be true. To judge is *ipso facto* to *know* that the content is true. What remains to be shown is how the information present in the conclusion can extend the information already known by the agent through the assertion of the premises.

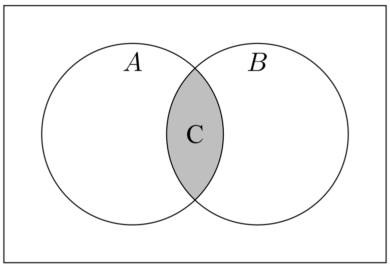
4 Fruitfulness and Concept Formation

Another piece of the Fregean puzzle of the informativity of logic is concept formation. In this regard, Frege is once again an anti-kantian. According to Frege, Kant's "too narrow view" of analyticity also stems from its too narrow view of concept formation. Kant's whole account of analyticity depends on seeing the constituents of a concept, which boils down to two options: either the predicate concept is already contained in the predicate concept, or it isn't. Given that Frege has a different conception of analyticity, he is expected to have a different idea about concepts and concept formation.

Frege claims that Kant

> seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, that is one of the least fruitful. But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it, cannot be inspected in advance; here we are not simply taking out of the box again what we have just put into it. The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic. The truth is that they are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house. (FREGE, 1953, §88)

Kant's way of defining concepts is well represented in Boolean algebra. Let C be composed of the concepts A and B. This means that $C = A \cap B$. In this case, the conclusions that 'C is A' or 'C is B' are trivial and could well be visualized with Venn's Diagrams:



This is what Frege meant in saying that conclusions are contained in the premises just as "beams are contained in a house". Under this scenario, analytic judgments are trivially obtained; consequently, logic cannot proceed fruitfully. Moreover, if the concepts A and B are such 'boundary lines', then the set of possible concepts formed from them is defined over all the boolean operations $A \cap B$, $A \cup B$, $A \setminus B$, and so on. Their conclusions are, in fact, already contained within such boundary lines.

The same point is made by Frege in the unpublished paper *Booles rechnende Logik und die Begriffsschrift*, now taking Boole's algebra of logic as the representative target. Using the same Venn diagram as the above, he argues that

> If we look at what we have in the diagrams, we notice that in both cases the boundary of the concept, whether it is one formed by logical multiplication or addition is made up of parts of the boundaries of the concepts already given. This holds for any concept formation that can be represented by the Boolean notation. [...] These really already contain the new concepts: all one has to do is to use the lines that are already there to demarcate complete surface areas in a new way. (FREGE, 1979, p.33-34)

Given that such concepts are formed by simple algebraic operations such as addition or multiplication, there is nothing one can conclude from then that was not already specified within the premises, just as beams are contained in a house. For Frege, this is the main reason ''[...] for the impression one easily gets in logic that for all our to-ing and fro-ing we never really leave the same spot'' (FREGE, 1979, p.34). Moreover, given that such procedure assumes ''logically perfect concepts'' as Frege would say, it is possible for a boolean logician to ''draws his inferences from the given assumptions by a mechanical process of computation'' (FREGE, 1979, p.35). Obviously, no new information could be derived if, in principle, there is a computable way to calculate possible outcomes of a given set of premises.

However, there is a more creative process for concept formation, labeled *fruitful*, that Frege defends to advance from such a sterile way of defining concepts. Up to 1884, when the majority of these comments were made, Frege had one major result to show in which this method was extensively used: his definition of the Ancestral Relation in section III of the *Begriffschrift*. The Ancestral is a definition for the transitive closure of a relation. Briefly, it is a logical analysis for total ordered relations, which by his time was something believed to be grounded in intuitions. The main result achieved, theorem 133, states that if a relation R is functional, then trichotomy holds for the ancestral of R.[22] From such a result, Frege argued, "[...] it can be seen that propositions which extend our knowledge can have analytic judgements for their content" (FREGE, 1953, §91). The same point is made in (FREGE, 1979, p.34), but now highlighting the fruitfulness of the definition:

> If we compare what we have here with the definitions contained in our examples [of the Ancestral], [...] totally new boundary lines are drawn by such definitions—and these are the scientifically fruitful ones. Here too, we use old

[22] One can imagine how to achieve this intuitively: if no two objects follow from the same object in R, then, for any two objects a and b, we just have to follow the chain of objects related through R and see that either b follows a, a follows b, or a and b are the same. But Frege's proof shows this result without such 'wandering' through the chain of objects. As he claims in (FREGE, 1967, §23), we can see in them how "[...] pure thought, irrespective of any content given by the senses or even by an intuition *a priori*, can, solely from the content that results from its own constitution, bring forth judgments that at first sight appear to be possible only on the basis of some intuition".

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> concepts to construct new ones, but in so doing we combine the old ones together in a variety of ways by means of the signs for generality, negation and the conditional.

It is common to find Frege speaking about both fruitful concepts and fruitful definitions. Definitions in the concept-script are presented with the aid of another special sign, the definition-stroke: $||\!\!-$. As he explains, the sign stipulates an identity of conceptual-content. Thus,

$$||\!\!- \alpha \equiv \beta$$

express the act of declaring β to have the same conceptual-content as α, where α is the *definiens* and β the *definiendum*. Again, this can be read as a speech act of a declaration, given that $||\!\!-$ functions as an illocutionary force indicating device.

All definitions expressed in the concept-script are nothing else than abbreviations: ''nothing follows from the proposition that could not also be inferred without it. Our sole purpose in introducing such definitions is to bring about an extrinsic simplification by stipulating an abbreviation'' (FREGE, 1967, §23). But this view clashes with Frege's remarks concerning the fruitfulness of definitions. For instance, in (FREGE, 1953, §70), he claims that ''Definitions show their worth by proving fruitful. Those that could just as well be omitted and leave no link missing in the chain of our proofs should be rejected as completely worthless.''[23] How can definitions be fruitful if they are non-creative and eliminable?

To answer this, Boddy (2021) hints at the fact that Frege has two conceptions of definitions: definitions *qua* explanation of concepts and definitions *qua* abbreviations in the proof system of the concept-script. This distinction, in her mind, should be enough to dissolve the above problem: when Frege speaks on definitions as fruitful, he is speaking about definitions *qua* explanations of concepts, and when he speaks of definitions as being non-creative and stipulative, he is speaking of definitions *qua* resources of the concept-script.

Following this idea, it is helpful to consider two levels of language in which an agent proceeds to make deductions with the concept-script. Frege did introduce such a distinction in the late unpublished paper *Logische Allgemeinheit* (FREGE, 1979, pp.258-262), between an expository language [*Darlegungsprache*] and an auxiliary language [*Hilfssprache*]. Although there's no explicit indication that Frege was speaking about the concept-script in making this distinction, the similarities are abundant, and it is worth applying them to the concept-script.

The distinction is motivated by the use-mention distinction that he had already introduced in the *Grundgesetze*. First, he mentions a language that serves as mediated way to grasp thoughts (or contents more generally). This language,

> [...] which I will call the *auxiliary language*[24] is to serve for us as a bridge

[23] See also Frege (2013, p.vi).

[24] This is the *Hilfssprache*, which the translators of the Posthumous Writings opt in translating as *object-language*, and *Darlegungsprache* as *meta-language*. They do so giving the resemblance to Tarski's use of the terms. They do not hold that both are the same, and the conclusion, as they claim, is left for the reader. Both translations are here changed, while the rest is left unaltered.

> from the perceptible to the imperceptible. [...] This auxiliary language is to be distinguished from the language in which I conduct my train of thought. That is the usual written or printed German, my expository language. But the sentences of the auxiliary language are the objects to be talked about in my expository language. (FREGE, 1979, p.260)

To bridge the imperceptible with the perceptible is to grasp contents by means of a written or spoken language. The similarities to the later distinction between object-language and metalanguage are significant, but Frege's distinction is not yet Tarskian. To begin with, there is no assumption concerning different levels of logicality, as the notions that the concept-script is supposed to formalize cannot be reduced to further notions: they are already in use in the expository language, albeit confusedly. What is left to do is just elucidations or explications [*Erläuterung*] and hints [*Winke*]. This corresponds to part I of *Begriffsschrift*, called "Definition of the Symbols" [*Erklärung der Bezeichnungen*], and the part I of *Grundgesetze*, the "Exposition of the Concept-script" [*Darlegung der Begriffsschrift*]. ''We must admit logically primitive elements that are indefinable. [...] Since definitions are not possible for primitive elements, something else must enter in. I call it explication [*Erläuterung*]'' (FREGE, 1984, p.300-301)

Thus, by taking it as a tool for reasoning, we shall assume that the concept-script is such an auxiliary language, given that grasping contents is a primary condition for judging truths, and German or English are expository languages, as they are used as expository means for the system. Putting this together with Boddy's account, definitions *qua* explanations are those performed in the expository language, while definitions *qua* resources with the concept-script are those performed in the auxiliary language. The connection between the two is precisely the idea of fruitfulness: it is by proving theorems within the auxiliary language that a definition can be deemed fruitful as an explanation of a concept within the expository language, as we shall see.

5 Decompositions

Fruitfulness, as it seems, must be understood pragmatically. A definition is said to be fruitful if it has ampliative consequences, *viz.* if it is useful in extending the boundary lines. And the way to properly show it is in the context of an inference. The proper role of logic is to draw inferences, and it is by doing so that we can recognize the truth of a proposition. Thus, it is by proving theorems that one can evaluate how fruitful, if at all, a definition is. But how can this be the case in the auxiliary language of the concept-script? The final piece is Frege's remarks on decompositions as means for concept formation. Recall that Kantian concepts were not capable of ampliative deductions precisely because of the way they are defined. Every Kantian concept F includes a list of characteristic notes $c_1, ..., c_n$ such that $F = c_1 \cap, ..., \cap c_n$. Conclusions from F, such as 'F is $c_{i \leq n}$' are in fact read '$F \subseteq c_{i \leq n}$', which will holds trivially. Since F is representable in Venn's diagrams, every conclusion is already contained in the definition.

Frege's alternative for concept formation stems directly from his conception of judgments and the priority given to them. He claims in multiple passages that "I start out from judgements and their contents, and not from concepts. [...] I only allow the formation of concepts to proceed from judgements" (FREGE, 1979, p.16). Similarly, in a letter to Anton Marty, he claims to think "[...] of a concept as having arisen by *decomposition* from a judgeable content" (FREGE, 1984, p.101, my emphasis). This is known as the *Priority Thesis*[25], *viz.,* that judgments are prior to concepts. Once we get a judgment, we can decompose from its main components. Consider the following Fregean example. From the arithmetical judgment

$$\vdash 2^4 = 16$$

one can decompose one or more of the constituents (numerals in object places or functional symbols for second-level functions) to obtain different functions. Precisely,

$$x^4 = 16, \quad 2^x = 16, \quad 2^4 = x, \quad x^y = 16, \quad ,2^x = y, \quad x^4 = y, \quad x^y = z$$

Each expression above can be regarded as different functions or relations obtainable from the original judgment: "4th root of 16", "logarithm of 16 to the base 2", "2 to the power of 4", and so on. Clearly, none of these concepts are constituents of the original judgment, asserting a simple identity property of the number 16. This is what Frege meant in saying that these consequences are contained in the premises as plants are contained in their seeds. As Frege explains, "[...] instead of putting a judgement together out of an individual as subject and an already previously formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of possible judgement" (FREGE, 1979, p.17).

The use of decomposition is closely related to the unrestricted comprehension principles in quantificational logic in the form:

$$\exists f \forall x (f(x) \leftrightarrow \varphi)^{26}$$

That is, for all x and any expression φ of the language, there is a function $f(x)$ that holds just in case the condition φ holds. Simply put, comprehension asserts that any well-formed expression of the language can define a concept, set, or in Frege's case, a function. We can see how decomposition provides comprehension principles in the language: given any decomposition φ of an expression φ' which is well-formed, there is a function f which holds whenever φ holds. Landini (2012, p.136) states that

> [...] decompositions provides comprehension principles as rich as those of a standard second-order calculus. As is well-known, a second-order calculus is not decidable and not even semantically complete. There can be no question as to the semantic informativeness of its theses. Decompositions is, therefore, all that is required for informativity.

[25] According to Ruffino (1991), the priority thesis can be read as an epistemological formulation of Frege's context principle, *i.e.*, ''Never to ask for the meaning of a word in isolation, but only in the context of a proposition'' (FREGE, 1953, p.x). In Ruffino's reading, this use of the context principle explains the Fregean version of the informativity thesis, as the priority thesis allows one to decompose contents and combine them into new concepts. Although Ruffino's characterization of decompositions in Frege's system is correct, it leaves out the problem of how to reconcile the fruitfulness of a definition with its intended non-creativity as simple stipulations, a problem that we try to address here.

[26] See Boolos (1985) and Landini (1996).

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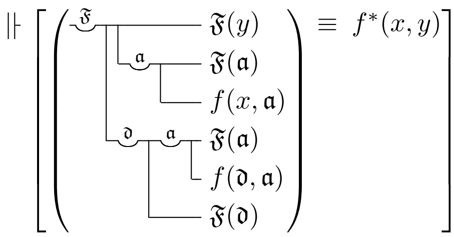
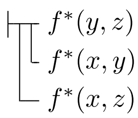
Following Landini, comprehension is the right path to understanding Frege's claims about the informativity of his logic as far as concept formation is concerned. The difference, also pointed out by him, is that Frege does not have comprehension principles but a rule of uniform substitution, given that the concept-script does not have a metatheory to include a comprehension schema. Instead, the system adopts Roman letters and function-markers to denote the generality of contents. If a judgment is true for a generic function, it is true for any.

Decompositions are essential in Frege's system, particularly in proving the basic facts about the ancestral definition (FREGE, 1967, §26). With the Ancestral, Frege argued for the possibility of proving theorems that, in his words, extend our knowledge. The definition, as an abbreviatory definition, is:



The *definiendum* on the right-hand side, $f^*(x, y)$, is read as "x follows y in the f-series". The definition was not originally called Ancestral. As the transitive closure of a relation, the definition is meant to grasp the concept of ''following in a sequence''. If f is interpreted as the parenthood relation P, then P^* is precisely the ancestral relation. For this reason, it is conventionally read as "x is the ancestral of y", *modulo* the relation f. Since stipulative definitions involve functional expressions, we can decompose either in the object or the function position to acquire new functions, either first or second-order, provided the priority thesis. Assuming the parenthood interpretation for f, we acquire the ancestral relation P^*, which allows us to decompose the *definiendum* in the second variable of the definition, acquiring a new function that reads "to be x's descendent", $[z : P^*(x, z)]$, or likewise in the first variable to obtain $[z : P^*(z, y)]$, which reads "being y's ancestor".

Practically, in the auxiliary language of the concept-script, it's easier to proceed with the function derived from the *definiendum*. From it, we can instantiate different axioms, deriving different theorems that are consequences of the definition provided. This was shown by Frege in section III of *Begriffsschrift*. Especially, theorems 98 and 133 are the two main results concerning the definition of the Ancestral. Theorem 98,



(98)

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shows that the definition is transitive. On the other hand, Theorem 133,

$$\vdash \begin{array}{l} (z \equiv y) \\ \neg\, f^*(z, y) \\ \neg\, f^*(y, z) \\ f^*(x, z) \\ f^*(x, y) \\ Fun(f) \end{array} \tag{133}$$

shows that the definition holds the trichotomy property. Above, $Fun(f)$ denotes the condition for f to be functional.[27] In this case, 133 means that if both y and z follow x from the series defined by f^*, then either y comes before z, z comes before y or both y and z are the same, hence the trichotomy property. These results provide higher-order properties that the definition is provable to satisfy, and they are proved with frequent use of decompositions through Frege's unstated rule of uniform substitution. Therefore, theorems 98 and 133 are facts *about* the definition, but they are not components of it.

To perform a decomposition, one necessarily goes beyond the auxiliary language in visualizing and deciding the intended function to acquire. In this regard, Frege's two-dimensional reading shows its convenience. This point motivates a different perspective on the subject, proposed by Danielle Macbeth, particularly in (2012). In her reading, Frege's concept-script must be read diagrammatically in such a way that the two-dimensional notation is as helpful in visualizing conceptual relations as the diagrams in Euclidean geometry are. As she claims, "Frege's concept-script enables an extension of our knowledge by revealing something new that is achieved by (literally) putting together in joining inferences parts of different wholes into new wholes" (*idem.,* p.311). Macbeth is correct in stressing the diagrammatic role that Frege's two-dimensional notation has in practice. The problem with her account is that it tends to reduce Frege's claims of informativity to a mere psychological realization. In fact, informativity stems from the intractability of second-order logic, something clearly related to Frege's adoption of decompositions. Thus, Macbeth's account seems only partially correct.

Concerning Frege's definitions in section III of the *Begriffsschrift*, she also argues that "if [...] we were to replace all the defined signs by their definitions, then we *would* have a mere theorem of logic", but, as she continues, "it would not be a theorem in the theory of sequences" (*idem,* p.306). Frege does claim that the definitions serve only practical roles: the proof's validity does not depend on the use of definitions, as we could achieve the same results solely with the *definiens*. And to be sure, the definitions occurring *in* the concept-script are only abbreviatory devices: $f^*(x, y)$ does not by itself say that x is the ancestral o y; it is just an abbreviation for the more complex function of the *definiens*. Although Frege is not always clear on the matter, we must keep in mind that when speaking about fruitful definitions, he is talking about definitions of *concepts*, that is, functions of the auxiliary language that are achieved by decomposition[28]. Macbeth is

[27] See Frege (1967, §31) for his precise definition.

[28] The fact that Frege always speaks of fruitful definitions in the context of rejecting Kant and Boole's definition of concepts speaks favorably to this interpretation.

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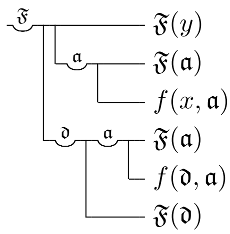
speaking about definitions as abbreviatory devices, and these are wholly eliminable and non-creative.

In that sense, Macbeth argues that theorems 98 and 133 would be different formulas if only the *definiens* were present. But the transformation from one to another does not carry information in the logical sense Frege is speaking of, but only in a psychological one. She claims that "the simple defined signs are needed if what is to be established is to be unequivocally about the concepts of interest. But their definitions are needed if anything about those concepts is to be established." (MACBETH, 2012, p.306). Again, this is only partially correct and can be made more precise if thought from the perspective of the auxiliary-expository language distinction. First, from the auxiliary language point of view, the analysis of the notion of "following in a sequence" has nothing to do with the choices of abbreviatory definitions. In fact, what matters is not the new sign being introduced (the *definiendum*), but the *definiens*. The proofs would hold equally without the definition. But from the expository language point, without the definition, the *definiens* would not be *about* ''following in a sequence''. In order to provide a sufficient analysis, both points should be connected. For instance, it is expected that the concept of following in a sequence must satisfy transitiveness (theorem 98) and trichotomy (theorem 133). But it is not the *definiendum* $f^*(x, y)$ that primarily satisfies these properties, but the *definiens*:



To understand this, here enters the expository language: by taking "y follows x in the f-sequence" as the analysis for the *definiendum* $f^*(x, y)$, Frege is connecting both points. In Boddy's terms, he is connecting the definition of the ancestral *qua* explanation of the concept of following in a sequence with the properties proved in the auxiliary language about the definition *qua* abbreviatory device. Nonetheless, it is only a proper analysis if the *definiens* is provable as being transitive and trichotomous. And *because* it is, Frege's analysis can be acceptable as sufficient for the desired concepts.[29]

On her diagrammatic reading, Macbeth claims that "Much as in reasoning through a diagram in Euclid one *perceptually* joins parts of different wholes to form new wholes, so here we *literally* join, by means of hypothetical syllogisms, parts of different wholes to form a new whole, ultimately, the whole that is theorem 133" (MACBETH, 2012, p.303). What she mentions as the hypothetical syllogisms are Frege's transitions from conditional judgment to conditional judgments by means of the rule of substitution and *modus ponens*. What she mentions as "parts of different wholes" are the decomposed functions that one obtains from other judgments. Her reading, nonetheless, offers an important insight: the definitions, much as Frege's notational choices, are indeed designed for visualizing these inferential patterns. The definitions,

[29] A similar point is made by Blanchette (2012, ch.1).

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as abbreviatory devices, are indeed helpful in visualizing the correct choices of functions one needs to instantiate in axioms or other judgments previously derived. As she says, "Much as reasoning through a diagram in Euclid does, such a course of reasoning realizes something new that had the potential to be derived but was in no way implicit in the starting point of the derivation" (*idem*, p.307).

To understand it, all decompositions are not performed in the auxiliary language,[30] they are not, to say it differently, steps in the language that are governed by purely logical laws.[31] For a judged expression to be decomposed, and the subsequent instantiation of any axiom with such acquired function, there seems to be a gap: nothing in the original expression says which part is to be decomposed or which function from a multiplicity is to be derived. This is a procedure to be checked and justified, but nonetheless, one that is not performed entirely in the logical language in question. If decompositions were mechanically and trivially determined, the fruitfulness of a definition wouldn't be a matter of testing one's consequences: "Definitions show their worth by *proving* fruitful" (FREGE, 1953, §70),[32] as Frege declared.

There is no mechanical procedure to decide such decompositions and their fruitfulness. This agrees with the undecidability of second-order logic. Such systems have advanced expressive power but limited meta-theoretical results. As a consequence, second-order logic is undecidable and semantically incomplete, thus, not reducible to a mechanical procedure of discovery. These meta-theoretical results must, by themselves, provide a negative answer to whether (second-order) logic is tautological. Propositional logic most certainly is since truth tables are mechanical procedures to determine whether a formula is a tautology. But even for propositional logic, the tautology problem is intractable, that is, not solvable in polynomial time,[33] meaning that, even though it is in principle computable, it is not practically feasible[34]. Second-order logic, in contrast, is not even in principle computable: there is no algorithm possible to compute, in general, if a second-order formula is a logical truth.

The question then is how, in this scenario, second-order logic can be considered analytical, as Frege would want. Kant would hardly accept Frege's proofs as analytical since decomposition and comprehension could be taken as intuitive features: what licenses us to accept and acknowledge the obvious existence of such functions or concepts declared by such principles, if not by some form of intuition?[35] Even if some form of intuition is a necessary condition for visualizing the existence of such functions (the ones acquired by decomposition), it is not clear how the logical validity of a proof featuring it would be affected, at least in the Fregean scheme of things. If we do accept the universalist reading of Frege's logic, then the realm

[30] Landini (2012) also argues in this direction.

[31] Still, any result following it is logically valid, since they are equivalent to a comprehension axiom that states that such acquired function exists.

[32] The emphasis is mine.

[33] See, for example, D'Agostino and Floridi (2008). This may help explain why Frege did not consider axiom (8) to be derivable from axioms (1) and (4). This particular case is still feasible and psychologically achievable, given that Łukasiewicz did manage to discover the dependency between the axioms. But the fact that propositional logic is intractable helps explain the gaps that we, as human agents, may show in the practice, just as Frege did. The dependency of axiom (8) on axioms (1) and (4) is nowhere as humanly evident as one would think.

[34] The difference is easily explained by D'Agostino (2016, p.176): "An effective procedure or "algorithm" by and large consists in a "mechanical method", *i.e.* one executable in principle by a machine, to solve a given class of problems (answering a certain class of questions). An effective procedure is *feasible* when it can also be carried out in practice, and not only in principle".

[35] Boolos (1985) was the one to mention such Kantian objection.

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of functions (or, likewise, concepts) is an objective fact. They are things that we *grasp*, never construct: "[...] a concept is something objective: we do not form it, nor does it form itself in us, but we seek to grasp it" (FREGE, 1984, p.133). Therefore, if every step in a proof is correctly judged as true, the instantiation of an axiom or theorem with some acquired functional expression is justified by the general character of such laws simply because, if the law holds generally, it holds for any given function. This answer puts weight on Frege's assumption about the objectivity of functions. Russell's Paradox, it is well known, is a fair objection that Frege took for granted that such a realm is easily obtainable. But at least in his own conception of analyticity, the picture is sound. Since in such proofs, decompositions are admissible and logically justified (simply because the existence of functions is a feature of the generality of logic), analytic judgments can extend our knowledge.

6 Logic as Science

Given that functions acquired through decompositions are not constituents of the premises that one starts with, they show real extensions of knowledge. And given that such inferences are logically and analytically valid in Frege's understanding, they are sound deductions. Thus, Frege can conclude that ''[...] propositions which extend our knowledge can have analytic judgements for their content.'' (FREGE, 1953, §91). Also, as Dummett (1991, p.42) likewise concludes:

> Deductive reasoning is thus in no way mechanical process [...]: it has a creative component, involving the apprehension of patterns[36] within the thoughts expressed [...]. Since it has this creative component, a knowledge of the premises of an inferential step does not entail a knowledge of the conclusion [...] and so deductive reasoning can yield new knowledge. Since the relevant patterns need to be discerned, such reasoning is fruitful; but, since they are there to be discerned, its validity is not called in question.

This questions Kant's original account of formal logic as a *cânon* of reason and opens up the old idea of formal logic as an *organon*[37]. Unquestionably, Frege viewed logic as a science, as the system of the concept-script was designed for the application of scientific research.

Even though logic (second-order logic, as Frege understands the matter) is not a computable endeavor (as the undecidability makes it clear), not just the proposition proved, but the whole proof is something new and not easily available in advance. Fregean decompositions, as shown, depend on the quantification theory of the concept-script, that deeply extends Boole's own logic. It is contrasting its own logic with Boole's that Frege Claims:

> Boolean formula-language only represents a part of our thinking; our thinking as a whole can never be coped with by a machine or replaced by purely mechanical activity. (FREGE, 1979, p.35)

[36]Such patterns, in Dummett's reading, are those functions that one can acquire by decomposition.
[37]On Kant's distinction between canon and organon, see (KANT, 1992, p.528)

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A similar passage appears in (FREGE, 1953, p.xvi): "It is possible, of course, to operate with figures mechanically, just as it is possible to speak like a parrot: but that hardly deserves the name of thought". Contrasting his own system of logic with mechanical machines is the Fregean way of saying that logic goes beyond what is previously given, just as a proper science would do.

But still, Frege could not be talking about the undecidability of second-order logic or any limited metatheoretical result when speaking about the informativity of logic. Landini's interpretation, then, is incomplete. At best, Frege was talking about a feature of second-order logic (decompositions), even though this feature is closely tied with its metatheoretical intractability as we now know (as second-order logic with full comprehension is incomplete). Thus, Frege's account of the informativity of logic, and its scientific status, should go beyond this feature.

For instance, Frege's proof theory wholly depends on the notion of a judgment. Since they are epistemic attitudes towards a content's truth, to carry over inferences is by itself to advance in an epistemic endeavor. It is not enough to derive a function through decomposition or an instance of the comprehension schema. Informativity is also achievable by deriving judgments about the definitions. In the concept-script, this follows the assertability condition for inferences: inferences are only performed from true premises to reach true conclusions. The scientific practice, that Frege wants the concept-script to be applicable to, is rooted in truth: "To discover truths is the task of all sciences" (FREGE, 1984, p.351), "the goal of scientific endeavour is truth" (FREGE, 1979, p.2). And thus, in logic, no step should be taken without the presence of the judgment-stroke, meaning that no deduction should proceed without every point being previously recognized as true: "[...] in presenting an inference, one must utter the premises with assertoric force, for the truth of the premises is essential to the correctness of the inference", Frege writes to Jourdain (FREGE, 1980, p.79). In fact, the judgment-stroke is, in Frege's mind, the ideal way to import the concept of truth to the system without adopting a truth predicate. This goes because Frege was not a Tarskian, and without a regimented hierarchy of formal languages, the truth predicate became circular (if not contradictory). This explains Frege's claims about the assertoric force of an assertion expressing the essence of logic, *viz.,* truth. As he claims in a posthumous paper: ''[...] the thing that indicates most clearly the essence of logic is the assertoric force with which a sentence is uttered'' (FREGE, 1979, p.52).[38]

Another feature of Frege's conception of logic as science can be highlighted through the distinction between auxiliary language and expository languages. Firstly, part of the scientific research is expected to be performed at the auxiliary level. There is no question that Frege's goal in presenting the concept-script is to be applicable to science in general,[39] as it was his hope that it would be used for such task, given that science proceeds through judgments. Secondly, the scientific practice also includes elucidations of concepts.[40] These are to be supplemented in the expository language in connection with the auxiliary one. Moreover, it is between both languages that fruitfulness is found, as we saw earlier. The very idea of

[38] On this topic, Greimann (2014) and Pedriali (2017) are good discussions.

[39] See, for instance, the preface to the *Begriffschrift*, or the paper *Über die wissenschaftliche Berechtigung einer Begriffsschrift* [*On the Scientific Justification of a Conceptual Notation*] in (FREGE, 1972, pp.83-89).

[40] Elucidations are also crucial for scientific practice, as a way of communicating one's system to others, as Frege himself did. See Weiner (1990, ch.6) on Frege's conception of elucidation. The connection between elucidations, the role of judgments in the scientific practice, and Frege's logic is also highlighted in Alnes (2018).

fruitfulness has a clear scientific motivation, as he says: ''[...] fruitfulness is the acid test of concepts, and scientific workshops the true field of study for logic'' (FREGE, 1979, p.33).

Finally, it is clear that Frege's decompositions are dependent on his treatment of generality in quantificational theory. In fact, in the background of such a theory lies a different logical understanding of quantification, one that is commonly called the universalist conception. This reading has started with Van Heijenoort (1967) seminal paper,[41] and it is antithetical with the modern model-theoretic one, as it assumes that logic has one single domain of interpretation, the universal domain. This allows Frege not only to decompose quantified expressions of the concept-script to acquire existent functions but also to instantiate such functions in axioms. In practice, the universal conception provides decompositions with the same strength of comprehension principles. But instead of schemas in the metalanguage, they allow decompositions to operate directly within the object-language, or the auxiliary language in Fregean terms.

In conclusion, Frege found a way to form new concepts out of judgments: by decomposition in the functional structure that they express. This far surpasses Kant's notion of concept formation and truly expands from what one has started with. But ultimately, informativity is found in the whole context of logical practice, which is not reducible to a mechanical procedure for Frege. These are judgments (and their expression as assertions), inferences, and definitions. Nonetheless, Frege's conception of logic as science is tied up with the informativity thesis: it is because logic can extend knowledge on its own that logicism, *i.e.*, the thesis that arithmetic is developed logic, can get off the ground.[42] Therefore, to the question of whether deductions can extend knowledge, Frege can clearly say yes, and reject the scandal of deduction.

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[41] We can find this view also in Hintikka and Sandu (1994), Ricketts (1986) and Goldfarb (2010).
[42] See May (2018) on this relation.

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