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VALIDITY, CONSISTENCY AND THE MEANING OF THE RECOVERY OPERATORS

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ABSTRACT: In this paper, we argue that the connective ∘ present in the Logics of Formal inconsistency (LFIs) are better understood as expressing the notion of *classicality*. We present the modal logic $\mathsf{LFI1}^{S0.5}$, whose modalities □ and ◊ captue he formal concepts of logical validity and logical consistency of the logic LFI1, the strongest LFI. We also prove that these modalities can also work as recovery operators.

Key-words: paraconsistency; consistency; modal logics; recovery operators; logics of formal inconsistency.

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1 Introduction

The idea of recovering classical inferences in non-classical logics matches well in the contemporary debate of Logical Pluralism. Given that classical logic is the logic standardly used in mathematical reasoning, it is convenient for a logical system to be capable of recovering classical inferences as far as possible. In such a point of view, a logician is not forced to give up all classical reasoning when she/he adopts a logic different from classical logic. Even in a non-pluralist point of view, there is sometimes the need for recapturing classical reasoning in non-classical logics. Priest (PRIEST, 2006), a upholder of dialetheism, recognizes that there are situations where Disjunctive Syllogism is valid, when no contradiction is involved.[1] Thus, such an inference rule is sometimes legitimate from a dialetheist point of view. However, as Antunes (ANTUNES, 2020) argues, Priest's proposal is not adequate, at least directly, to recover classical inferences because of the weak expressiveness of the *logic of paradox* (LP) (ASENJO, 1966; PRIEST, 1979), the system widely defended by Priest.

In the logical literature, we can find different ways of representing classical inferences in non-classical logics. One of the most known recovery strategies comes from intuitionistic logic. Kolmogorov's (USPENSKY, 1992), Gödel's (GÖDEL, 1933) and Glivenko's (GLIVENKO, 1929) translations of intuitionistic logic into classical logic provide a way of representing classical logic in intuitionistic logic by means of double negation.

The development of paraconsistent logics, mainly leaded by the Brazilian (COSTA, 1974) and Belgian (BATENS, 2000) schools of paraconsistency, inaugurated a trend in the logical literature of introducing an operator in the object language, which is able to recover classical inferences once some 'consistency assumptions' are made. In paraconsistent logics (CARNIELLI; CONIGLIO; MARCOS, 2007), such connectives *recover* the explosive character of propositions. In the case of paracomplete logics (MARCOS, 2005b), they recover the determinedness of propositions. And in the paranormal case (OMORI, 2020), they recover both explosiveness and determinedness. Because these connectives are able

[1] Dialetheism is the ontological thesis that asserts the existence of contradictions in reality. For a nice presentation of this thesis, we invite the reader to read Priest, Berto, and Weber (2022).

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to recover properties commonly lost in the departure from classical logic, they are called *recovery operators*. In sum, recovery operators are tools to recover inferences we lose when departing from classical logic. It is common to see them being understood as incorporating metatheoretical concepts in the object language of logic.

Recovery operators were vastly investigated from a technical point of view, and they are not new in philosophical discussions. For example, Bochvar (BOCHVAR; BERGMANN, 1981) introduces these connectives to distinguish paradoxical sentences from non-paradoxical ones. Halldén (HALLDÉN, 1949), Åqvist (ÅQVIST, 1962) and Segerberg (SEGERBERG, 1965) introduce them to distinguish meaningful from meaningless sentences.[2] In the study of truth-theories based on many-valued logics we find interesting applications of recovery operators. Also, in this research program, the application of these operators helps to distinguish paradoxical sentences (like *liar sentence*) from non-paradoxical sentences.[3] [4]. Although the point of introducing such connectives is somewhat clear, their informal interpretation is still an open problem. For example, in the case of *Logics of Formal Inconsistency* (LFIs) (CARNIELLI; CONIGLIO; MARCOS, 2007), Ferguson (FERGUSON, 2018, p.1) argues that "the notion of consistency is too broad to draw decisive conclusions with respect to the validity of many theses involving the consistency connective." In this work, we will advance some critics about the metatheoretical interpretation of the consistency operator $\circ$ present in LFIs and argue that this connective can have a nice metatheoretical interpretation, compatible with the semantic conditions of LFI1. In the Section 2, we present the logic LFI1 and its informal interpretation. Although we show that the connective $\circ$ does not stand for metatheoretical consistency, we argue that $\circ$ captures the concept of classicality. In the Section 3, we present the modal logic $\mathsf{LFI1}^{S0.5}$, whose modalities $\Box$ and $\Diamond$ capture the formal concepts of logical validity and logical consistency of LFI1, respectively. In the Section 4, we show that $\mathsf{LFI1}^{S0.5}$ is capable to define modalities that recover classical inferences. In Section

[2]In (RESCHER, 1969) and (BOLC; BOROWIK, 2013), one finds nice surveys of many-valued logics with different applications.

[3]For such applications, we invite the reader to confer the works (BARRIO, E. A.; PAILOS, F. M.; SZMUC, D. E., 2017), (BARRIO, E.; PAILOS, F.; SZMUC, D., 2016), where Barrio et al. show some non-trivial truth-theories with a transparent truth predicate.

[4]We refer the reader to (CORBALÁN, 2012) for a wide investigation of such connectives.

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5, we close the discussion with a few remarks. In the Appendix 5, we present the proof system for $\mathsf{LFI1}^{\mathsf{S0.5}}$.

2 Consistency from a paraconsistent point of view

Provability logics (BOOLOS, 1995) has showed that the incorporation of metatheoretical concepts in the object language of the logic provides an interesting analysis of formal concepts via logical systems. In recent years, many works in this direction were done and the LFIs became widely known in the literature. LFIs are paraconsistent logics which respect a strict version of the principle of explosion, called *gentle principle of explosion*.

$$\circ\varphi, \varphi, \neg\varphi \vdash \tag{1}$$

Where $\circ\varphi$ means that "φ is consistent". That is, the principle of explosion is restricted to consistent formulas. In what follows, we present a LFI system which was proposed to capture the concept of inconsistency, the logic LFI1 (CARNIELLI; MARCOS; DE AMO, 2004). LFI1 takes the connective $\bullet$ of inconsistency as primitive instead of $\circ$. But, as we will see, $\circ$ is definable in terms of $\bullet$ and negation.

Definition 2.1. *The language of* LFI1 *is* $\mathcal{L}^{\bullet} = \{\mathcal{V}, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \bullet\}$, *where* $\mathcal{V} = \{p_i | i \in \mathbb{N}\}$ *is the set of propositional variables, the connectives* $c \in \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ *are interpreted as usual, and* $\bullet$ *is a unary connective of inconsistency. The set of formulas of* LFI1, $For(\mathcal{L}^{\bullet})$ *is inductively defined as:*

$$p_i \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \bullet\varphi \mid \varphi \leftrightarrow$$

For $\varphi, \psi \in For(\mathcal{L}^{\bullet})$. $\bullet\varphi$ *means that* φ *is inconsistent.*

The logic LFI1 *is characterized by the matrix* $M_{\mathsf{LFI1}} = \langle\{1, \frac{1}{2}, 0\}, \bullet, \neg, \wedge, \vee, \rightarrow, \{1, \frac{1}{2}\}\rangle$, *whose operations have the following truth-tables:*

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	1	1	1

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

	$\neg$	$\bullet$
1	0	0
$\frac{1}{2}$	$\frac{1}{2}$	1
0	1	0

A valuation v *of* LFI1 *is a function* $v : For(\mathcal{L}^{\bullet}) \to \{1, \frac{1}{2}, 0\}$ *such that:*

1. $v(\neg\varphi) = 1 - v(\varphi)$;
2. $v(\bullet\varphi) = 1 - [2v(\varphi) - 1]$;
3. $v(\varphi \vee \psi) = max\{v(\varphi), v(\psi)\}$.

The set of all valuations v *is called* sem_{LFI1}. *Let* $\varphi \in For(\mathcal{L}^{\bullet})$ *be a formula. We say that* φ *is a tautology (satisfiable/consistent) iff for all (some)* $v \in sem_{\mathsf{LFI1}}$, $v(\varphi) \in \{1, \frac{1}{2}\}$. *If* $v(\varphi) = 0$ *for every* $v \in sem_{\mathsf{LFI1}}$, *then* φ *is a* contradiction *of* LFI1. *The* semantic consequence relation, $\models_{\mathsf{LFI1}} \subseteq \wp(For(\mathcal{L}^{\bullet})) \times For(\mathcal{L}^{\bullet})$ *is defined as usual. Consider* $\Gamma \cup \{\alpha\} \subseteq For(\mathcal{L}^{\bullet})$, *then we say that* α *is a* semantic consequence of Γ *(*$\Gamma \models_{\mathsf{LFI1}} \alpha$*) iff, for every* v, *if* $v(\gamma) \in \{1, \frac{1}{2}\}$, *for every* $\gamma \in \Gamma$, *then* $v(\alpha) \in \{1, \frac{1}{2}\}$.

From these connectives, we define the consistency connective $\circ\varphi$ as:

$$\circ\varphi \equiv \neg \bullet \varphi$$

As a consequence, $\circ\varphi$ has the following truth-table:

	$\circ$
1	1
$\frac{1}{2}$	0
0	1

For the moment, we will not present the proof system for LFI1. In the Appendix we will present a sound and complete labelled tableaux for the modal extension of this system, in the lines of (CARNIELLI, W. A., 1987; BEZERRA, 2021). Although LFI is a label for a plethora of formal systems, our choice for LFI1 stems from its truth-functionality and deductive strength. In general, LFIs have the interesting property of recapturing classical inferences due to *Derivability Adjustment Theorem* (DAT).[5] The next proposition shows some interesting applications of DAT.

[5]This theorem can be checked in (CARNIELLI; CONIGLIO; MARCOS, 2007). In Antunes (ANTUNES, 2020), one finds a version of DAT theorem for LFI1.

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Proposition 2.2. *The following items hold for* LFI1*:*[6]

1. $p \vee q, \neg p \nvDash q$;
2. $\circ p, \circ q, p \vee q, \neg p \vDash q$;
3. $p \to q, \neg q \nvDash \neg p$;
4. $\circ p, \circ q, p \to q, \neg q \vDash \neg p$;
5. $p \to q \nvDash \neg q \to \neg p$;
6. $\circ p, \circ q, p \to q \vDash \neg q \to \neg p$.

In general, LFIs are intended to be a logical basis for non-trivial inconsistent theories, thus being able to deal with contradictory reasoning in a "remarkably natural and elegant way", as Carnielli (CARNIELLI, W., 2011) highlights, since they are capable to separate consistent formulas from inconsistent ones with the aid of the connective $\circ$. Diverging from traditional approaches to contradictions, which defends their existence in the real world, such as Priest's *Dialetheism*, LFIs face contradictions in an epistemological perspective as the following passage shows:[7]

> At this point we would like to call attention to the fact that logics of formal inconsistency, although neutral with respect to real contradictions, are perfectly well suited to the idea that we do not know whether or not there are real contradictions, despite the fact that we have to deal with contradictions. When a physicist considers two theories to be inconsistent when put together, (s)he is doing exactly the kind of thing that logics of formal inconsistency are designed for – using classical logic in the theories taken separately, but restricting the principle of explosion with respect to the contradiction yielded by combining them together. Thus we affirm that *logics of formal inconsistency act primarily within the epistemic domain of logic, without any commitment to the existence of real contradictions*. (CARNIELLI; RODRIGUES, 2012, p.13)

[6]We will omit the subscript LFI1 in $\vDash$ whenever the context is clear.
[7]For an ontological defense of paraconsistency, check (PRIEST, 2006) for more details.

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This epistemological approach to contradictions is interesting for two reasons. First, it is certainly more scientifically guided and philosophically plausible to accept that contradictions occur only in the level of information than accepting that there is a real contradiction. The second reason is that, in general, paraconsistent logics are deductively weak to represent many forms of reasoning present in mathematics whose validity are dificult to neglect. For example, some paraconsistent logics do not validate modus ponens as well as contraposition, which are forms of reasoning present in mathematical reasoning. But, differently from these paraconsistent logics, LFIs are able to recover these inference rules by estipulating the consistency of these sentences. Then, these logics do not put ourselves distant from the classical reasoning, standardly used in Mathematics. In sum, this approach can shed new light on the nature of contradictions.

2.1 Some critics on the interpretation of $\circ$

This broad character of $\circ$ in LFIs makes it difficult to say what aspect of consistency it captures, even in the epistemological realm. For us, it is clear that the proposed meaning for $\circ$ does not capture the metatheoretical concept of consistency.[8] In order to show that $\circ$ does not capture metatheoretical consistency, we will give an indirect argument. Instead, we will give a squeezing argument for LFI1 *a la* Kreisel (KREISEL, 1967), and then we will compare the informal notion of the argument with the proposed informal reading of $\circ$. The argument runs as follows. Let D_{LFI1} and V_{LFI1} respectively denote validity in a formal deductive system for a LFI1 and validity in an adequate model-theory. Lastly, let Val_{Des} be defined as follows:

$Val_{Des}(\varphi)$: φ is designated in all models.

The informality of Val_{Des} lies in the non specification of the cardinality of the set of truth-values V of the matrix. Indeed, Val_{Des} may stand as an informal notion of validity for many-valued logics, in general. And, of course, we have to take for granted that

[8] In (MENDONÇA; CARNIELLI, 2020), Mendonça & Carnielli recognize that the operator $\circ$ does not faithfully capture model-theoretical consistency. In their paper, they argue that the operator $\circ$ captures a particular form of consistency, present in information theories.

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Val_{Des} adapts to the recursive definitions of the valuations of LFI1. It is clear that every LFI1-theorem is informally valid. Then (1) $D_{\mathsf{LFI1}}(\varphi) \Rightarrow Val_{Des}(\varphi)$. Moreover, if φ is designated in all models, φ is designated in the three-valued matrices of LFI1. Then, (2) $Val_{Des}(\varphi) \Rightarrow V_{\mathsf{LFI1}}(\varphi)$. The squeezing argument runs as follows:

Argument 2.3. *The squeezing argument for* LFI1 *can be summarized as follows:*

(1)	$D_{\mathsf{LFI1}}(\varphi) \Rightarrow Val_{Des}(\varphi)$	*(1)*
(2)	$Val_{Des}(\varphi) \Rightarrow V_{\mathsf{LFI1}}(\varphi)$	*(2)*
(3)	$V_{\mathsf{LFI1}}(\varphi) \Rightarrow D_{\mathsf{LFI1}}(\varphi)$	*Completeness Theorem*
(4)	$V_{\mathsf{LFI1}}(\varphi) \Rightarrow Val_{Des}(\varphi)$	*Logic (1),(3)*
(5)	$D_{\mathsf{LFI1}}(\varphi) \Leftrightarrow Val_{Des}(\varphi) \Leftrightarrow V_{\mathsf{LFI1}}(\varphi)$	*Logic (1)-(4)*

Argument 2.3 establishes that the informal notion Val_{Des} extensionally collapses with the formal notions D_{LFI1} and V_{LFI1} when LFI1-formulas are at issue. Consider the dual of $Val_{Des}(\varphi)$, defined as

$Con_{Des}(\varphi)$: φ is designated in some model.

Now we present some remarkable differences between Con_{Des} and $\circ$. Consider the following (informal) assertion:

Assertion 2.4. *Let $\wedge$ and $\to$ be interpreted as the in truth-tables of* LFI1*. Then,*

$$(Con_{Des}(\varphi) \wedge Con_{Des}(\neg\varphi)) \to Con_{Des}(\varphi \wedge \neg\varphi) \tag{2}$$

is not valid.

Proof. Consider the following stance of the meta schema (2), where $\varphi = \circ p$. Now take two models M and M' which respectively attribute 1 and $\frac{1}{2}$ to p. Then $Con_{Des}(\circ p)$ and $Con_{Des}(\neg \circ p)$ are true, and hence $Con_{Des}(\circ p) \wedge Con_{Des}(\neg \circ p)$ is true. But $\circ p \wedge \neg \circ p$ is false in all models. Therefore, $Con_{Des}(\circ p \wedge \neg \circ p)$ cannot be true.

Q.E.D.

On the other hand, the following formulas are LFI1 theorems:

1 $\vdash_{\mathsf{LFI1}} (\circ p \wedge \circ\neg p) \to \circ(p \wedge \neg p)$;

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2 $\vdash_{\mathsf{LFI1}} \circ\bot$.

Then, as Assertion 2.4 shows, the informal notion of consistency expressed by Con_{Des} does not coincide with the notion of consistency expressed by $\circ$. Since Val_{Des} is an informal bridge between D_{LFI1} and V_{LFI1}, we can conclude that $\circ$ is independent from model-theoretical and proof-theoretical definitions of consistency. Even so, we will show that the operator $\circ$ of LFI1 can receive an even more general metalogical interpretation.

Now, one might reasonably ask: what about others LFIs? The Assertion 2 holds for LFI1 because it validates:

$$(\circ\varphi \wedge \neg \circ \varphi) \leftrightarrow \bot \tag{3}$$

In LFI's where schema (3) holds, the independence between $\circ$ and its formal reading may not be true. But many of these logics, such as the minimal LFI mbC, do not have theorems of the form $\circ\varphi$. That is, they cannot say that their own theorems are consistent in the sense of $\circ$. There are others that do validate the metaschema of Assertion 2 but do not (3). But many of them do not validate theorems of the form $\circ\varphi$.

Our conclusion is compatible with some proponents of the LFIs about $\circ$, who defend that:

> Taking into account that a primitive concept is one that is not defined in terms of other concepts, the idea of consistency viewed as a primitive concept is rendered formal by means of a propositional operator (or a primitive connective) governed by certain logic axioms.
>
> Consistency, in this sense, would certainly be a notion totally independent of model theoretical and proof-theoretical means. (BUENO-SOLER; CARNIELLI, 2017, p. 12)

So, according to them, consistency should be taken as a primitive concept and it its meaning should be given by the axioms it satisfies also in relation with the other connectives of the logical language. The problem is that even if one defends that the meaning of the logical connectives is particular to the logical systems, in weak LFIs the axioms involving $\circ$ are too broad to be interpreted as consistency, as Ferguson (FERGUSON, 2018) argues.

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And in stronger LFIs we have the problem of the principles involving $\circ$ which are not compatible with its intended interpretation.

On the other hand, the point of introducing this kind of operator in paraconsistent logics is to label sentences whose behaviour is classical, i.e. consistent. Many LFIs scholars defend that every formula tagged by $\circ$ is non-explosive. But explosiveness has a formal meaning. Thus it is difficult to say that the consistency expressed by $\circ$ is independent of model-theoretical and proof-theoretical approaches. So, if the informal interpretation is not independent from its formal approaches, it is also difficult to defend that the consistency expressed by $\circ$ is primitive. As argued in recent works such as in (SMILEY, 1998), (SMITH, 2011), (GRIFFITHS, 2014), (HALBACH, 2020) and (BEZERRA; VENTURI, 2021), the informal notions captured by formal notions logical consequence are not primitive concepts. Instead, they are a result of a conceptual sharpening without which it is not possible to say that these informal notions correspond to their formal counterparts. But, as we argued before, it is not clear how to interpret $\circ$ in formal terms as standing for consistency.[9]

In our view, metatheoretical notions of validity and consistency have a quantificational component in their definition. We say that a body of statements C is consistent if *there is no* statements A such that A & non-A in C. We say that a statement φ is valid (resp., consistent) iff φ is true in *every* (resp., some) model. The matricial semantics for LFI1 is not able to capture such subtleties.[10] For this reason, we do not think that the characterization of such notions via truth-tables is the right way. The modal approach to these concepts shows to be more promising.

Although we have presented some problems concerning the informal interpretation of $\circ$, it is clear that $\circ$ echoes some metatheoretical notion. Even if such connective validates some counter-intuitive principles about consistency, it is a nice device to label formulas which behaves like in classical logic. If it were not the case, the theorem DAT might not work. We think that the best interpretation for LFI1's $\circ$ goes in direction of Omori's classicality interpretation (OMORI, 2020). That is, $\circ\varphi$ is interpreted as "φ has a

[9]Moreover, it is not difficult to see LFIs being defined as logics which internalize a metatheoretical notion of consistency, as one can find in works such as Carnielli & Fuenmayor's paper (CARNIELLI; FUENMAYOR, 2020), where they defend that Gödel's incompleteness results depend on the consistency of some formulas occurring in the proofs of Gödel's theorems.

[10]This also applies to the *valuation semantics* that also characterizes this logic.

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classical value." This interpretation is more general than consistency, because a formula can be classically false under all intepretations (then inconsistent) and a formula can have a designated value (i.e., satisfiable/consistent) whithout having a classical value. Then, classicality and consistency are independent concepts. It is easily seen that such interpretation makes sense of LFI1's validities. In particular, it makes sense of the formulas 1 and 2. For the formula 1, if p and $\neg p$ have classical values, then $p \wedge \neg p$ has a classical value, since LFI1 is a subsystem of Classical Propositional Logic (CPL). For the formula 2, it is clear that $\bot$ always has the classical value 0. Then, $\bot$ has a classical value.

In what concerns the intepretation of $\circ$ belonging to the others LFI's, we will leave this question as an open problem. The reason to leave this question open is simple: the meaning of the connectives is local. That is, we have to analyse at least the inferential behavior of $\circ$ in the particular LFI L before fixing an informal interpretation. That is, we have to analyse how $\circ$ behaves before $\vdash_{\mathsf{L}}$ (and $\models_{\mathsf{L}}$). For example, it is difficult to defend that the connective $\circ$ in mbC (CARNIELLI; CONIGLIO; MARCOS, 2007), the weakest LFI, since the only axiom and inference schema which involve $\circ$ are $\circ\varphi \to (\varphi \to (\neg\varphi \to \psi))$ and $\circ\varphi, \varphi, \neg\varphi \vdash_{\mathsf{mbC}} \psi$. They alone are not sufficient to characterize the meaning of consistency and that of classicality.

The reasoning raised in the last paragraph applies to all logical systems. In order to assert that a connective $\clubsuit$ has a particular informal interpretation, one should first look at the axioms or the inferential principles governing $\clubsuit$. Without such interpretation exercise, any discussion on the meaning of $\clubsuit$ will be a pointless verbal dispute.[11].

[11] Modal logics are paradigmatic cases. There are modal logics which are difficult to fix informal interpretation. The axioms and rules which govern the basic logic K (HUGHES; CRESSWELL, 1996) are too wide in such a way that they are compatible with several interpretations. The axiom K and the rule Nec are compatible with provability, alethic, epistemic, deontic and temporal interpretations of the modalities $\Box$ and $\Diamond$. But, the logic K itself is too general to be interpreted. On the other hand, there are modal systems which are interpreted in more than one way. Such is the case with the modal logic S4.2, which has at least two informal interpretations: epistemic (STALNAKER, 2006) and set-theoretical (HAMKINS; LÖWE, 2008)

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3 A modal approach to consistency

In this section, we present the modal logic $\mathsf{LFI1}^{S0.5}$, whose modalities $\Box$ and $\Diamond$ capture the formal concepts of logical validity and logical consistency, respectively.

Definition 3.1. *The modal extension of the language* $\mathcal{L}^{\bullet}$ *is defined as* $\mathcal{L}^{\bullet}_{\Box\Diamond} = \mathcal{L}^{\bullet} \cup \{\Box, \Diamond\}$. *The set of formulas* $For(\mathcal{L}^{\bullet}_{\Box\Diamond})$ *is generated as usual.*

Definition 3.2. *Let* M_{LFI1} *be the matrix of* LFI1 *defined as in Definition 2.1. An* M_{LFI1}*-modal model is a structure of the form* $\mathcal{M}_{\mathsf{LFI1}} = \langle W, N, R, v\rangle$ *where* W *is a set of worlds,* $N \subseteq W$ *is a set of normal worlds,* R *is a reflexive relation on* N *such that if* $y \in W$ *and* $x \in N$*, then* xRy*, and* v *is an assignment such that for every* $w \in W$, $v_w(p) \in \{1, \frac{1}{2}, 0\}$. *The function* v *is recursively extended as in Definition 2.1 for the connectives that are not modalities:*

1 *The truth conditions of* $\neg\varphi$, $\varphi \vee \psi$, $\varphi \to \psi$ *and* $\bullet\varphi$ *at* w *are given by the truth-tables of* LFI1.

The interpretation of the modal operators runs as follows:

For $w \in N$*:*

2 $v_w(\Box\varphi) = 1$ *iff for all* $y \in W$ *such that* wRy, $v_y(\varphi) \in \{1, \frac{1}{2}\}$*; otherwise* $v_w(\Box\varphi) = 0$*;*

3 $v_w(\Diamond\varphi) = 1$ *iff for some* $y \in W$ *such that* wRy, $v_y(\varphi) \in \{1, \frac{1}{2}\}$*; otherwise* $v_w(\Diamond\varphi) = 0$*;*

4 *For* $w \notin N$*: the value* $v_w(M\varphi)$ *is arbitrary in* $\{1, \frac{1}{2}, 0\}$ *for* $M \in \{\Box, \Diamond\}$.

Let $\varphi \in For(\mathcal{L}^{\bullet}_{\Box\Diamond})$ *be a formula of the language* $\mathcal{L}^{\bullet}_{\Box\Diamond}$. φ *is* true *in a* M_{LFI1}*-modal model* $\mathcal{M}$ *iff for every* $w \in N$, $v_w(\varphi) \in \{1, \frac{1}{2}\}$. φ *is* valid *iff it is true in every* M_{LFI1}*-modal model. The relation* $\models_{\mathsf{LFI1}^{S0.5}}$ *is defined in a similar way as Definition 2.1 for worlds* $w \in N$.

Definition 3.3. *The* modal counterpart of LFI1*, that we indicate by* $\mathsf{LFI1}^{S0.5}$*, is the modal logic in the language* $\mathcal{L}^{\bullet}_{\Box\Diamond}$ *that consists of all* M_{LFI1}*-valid formulas.*

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The modalities introduced in Definition 3.2 were first introduced by Lemmon (LEMMON, 1957) in the context of CPL by the name S0.5. Scotch et al. (SCOTCH et al., 1978) investigate bivalent modalities in the context of $_3$ (RESCHER, 1969).[12] The intended interpretation of the modalities presented in Definition 3.2 is formal: $\Box\varphi$ and $\Diamond\varphi$ intend to mean, respectively, that "φ is a LFI1 tautology" and "φ is consistent/satisfiable in LFI1." In (BEZERRA, 2021; BEZERRA; VENTURI, 2022) these modalities are studied for a large class of finite many-valued logics L, and in (BEZERRA, 2021) is provided a sound and complete proof systems for these many-valued modal logics. There, it is also proved that such modal logics have a well-justified formal interpretation. Let $\mathsf{L}^{\mathsf{S0.5}}$ be the modal modal counterpart of L. In (BEZERRA; VENTURI, 2022), it is proved that $\Box\varphi$ is valid in $\mathsf{L}^{\mathsf{S0.5}}$ if and only if φ is a tautology of L. Clearly, such results cover $\mathsf{LFI1}^{\mathsf{S0.5}}$. Then, by the results of (BEZERRA; VENTURI, 2022), $\Box\varphi$ is valid in $\mathsf{LFI1}^{\mathsf{S0.5}}$ if and only if φ is valid in LFI1. In this sense, $\mathsf{LFI1}^{\mathsf{S0.5}}$ is a validity theory for the logic LFI1. The bivalence of $\Box$ and $\Diamond$ makes sense when interpreted from a metatheoretical point of view. If the metatheory of MVLs is classical, then statements like "φ is valid/consistent" only receive classical values.

In CPL all tautologies are equivalent each other. This does not happen with LFI1 as we will show below.[13] Consider the following conditional:

$$\varphi \supset \quad := \quad (\varphi \to \psi) \wedge (\neg\psi \to \neg\varphi)$$

Its interpretation is given by the following truth-table:

$\supset$	1	$\frac{1}{2}$	0
1	1	0	0
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	1	1	1

[12] In their paper, Scotch et al. investigate bivalent modalities in over reflexive and transitive frames, and frames where the accessibility relation is an equivalence relation. It is worth to note that they not consider a subset of normal worlds.

[13] In general, this can be said about several many-valued logics. In these logics, contradictions are not equivalent each other or tautologies are not equivalent each other.

$\supset$ is the conditional of the logic RM_3 (ANDERSON; BELNAP, 1975). By the semantic definition of $\circ$ we know that, for any φ, $\circ\circ\varphi$ is a tautology of LFI1. So:

Theorem 3.4. $\nvDash_{\mathsf{LFI1}} \circ\circ p \supset \neg(p \wedge \neg p)$.

Proof. Consider a valuation $v \in sem_{\mathsf{LFI1}}$ such that $v(p) = \frac{1}{2}$. Then $v(\circ\circ p) = 1$ and $v(\neg(p \wedge \neg p)) = \frac{1}{2}$. By definition of $\supset$, we obtain $v(\circ\circ p \supset \neg(p \wedge \neg p)) = 0$. Q.E.D.

So, by the Theorem 3.4, we have that not all tautologies are equivalent each other. So, if $\mathsf{LFI1}^{\mathsf{S0.5}}$ is a validity theory for LFI1, $\mathsf{LFI1}^{\mathsf{S0.5}}$ should be able to express this non-equivalence. Consider the following abbreviation:

$$\blacksquare\varphi \quad := \quad \neg\Diamond\neg\varphi$$

Then, its semantic clause is stated as follows:

Definition 3.5. *Let $\mathcal{M} = \langle W, N, R, v\rangle$ be a M_{LFI1}-modal model and $w \in N$.*

$v_w(\blacksquare\varphi) = 1$ *iff for all $y \in W$ such that wRy, $v_y(\varphi) = 1$; otehwise, $v_w(\blacksquare\varphi) = 0$*

The modality $\blacksquare$ expresses tautologicity in a very narrow sense. Only formulas that receives 1 under every valuations are tautologies in $\blacksquare$ sense. Given Definition 3.5, we can see that if a formula is valid in $\Box$ sense, it may not be valid in $\blacksquare$ sense, as the following theorem shows:

Theorem 3.6. $\Box\varphi \to \blacksquare\varphi$ *is not valid in* $\mathsf{LFI1}^{\mathsf{S0.5}}$.

Proof. Let $\mathcal{M} = \langle W, N, R, v\rangle$ be a M_{LFI1}-modal model such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w,w),(w,y)\}$ and $v_w(\varphi) = 1$ and $v_y(\varphi) = \frac{1}{2}$. Then, $v_w(\Box\varphi) = 1$ and $v_w(\blacksquare\varphi) = 0$. Therefore, $v_w(\Box\varphi \to \blacksquare\varphi) = 0$. Q.E.D.

The following result shows some relations between $\blacksquare$, $\Box$ and $\Diamond$:

Theorem 3.7. *The following formulas are valid in* $\mathsf{LFI1}^{\mathsf{S0.5}}$*:*

1. $\blacksquare(\varphi \to \psi) \to (\blacksquare\varphi \to \blacksquare\psi)$*;*

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2. $\blacksquare\varphi \to \varphi$;

3. $\blacksquare\varphi \to \Diamond\varphi$;

4. $\blacksquare\varphi \to \Box\varphi$;

5. $\blacksquare(\varphi \wedge \psi) \leftrightarrow (\blacksquare\varphi \wedge \blacksquare\psi)$.

In the next section we will show that the modalities $\Box$, $\Diamond$ and $\blacksquare$ can also be used as interesting tools in the study of recovery operators.[14]

4 Validity, consistency and recovery operators

Now we will argue that the modalities investigated in the latter section can also be used as recovery operators. They will allow to recover all the inferences that we lose when we go from S0.5 to $\mathsf{LFI1}^{\mathsf{S0.5}}$. After presenting DAT results for $\mathsf{LFI1}^{\mathsf{S0.5}}$, we will present a modality that intends to capture the basic ideas of the classicality operator $\circ$, in order to give a broader investigation of metatheoretical concepts which $\mathsf{LFI1}^{\mathsf{S0.5}}$ is able to represent.

Given the operator $\blacksquare$, we define the following modality:

$$\blacktriangle\varphi \quad := \quad \blacksquare\varphi \vee \blacksquare\neg\varphi$$

Therefore, the truth condition of $\blacktriangle\varphi$ is given by the following clause:

Definition 4.1. *Let $\mathcal{M} = \langle W, N, R, v\rangle$ be a M_{LFI1}-modal model and $w \in N$.*

$v_w(\blacktriangle\varphi) = 1$ *iff (for every $y \in W$ such that wRy, $v_y(\varphi) = 1$) or (for every $z \in W$ such that wRz, $v_z(\varphi) = 0$); otherwise,* $v_w(\blacktriangle\varphi) = 0$.

[14]We also refer the reader to Coniglio and Peron's paper (CONIGLIO; PERON, 2013), where they investigate subclassical fragments of the logic $\mathsf{S0.5}^0$, that is Lemmon's classical $\mathsf{S0.5}^0$ without axiom $\Box\varphi \to \varphi$. There they show that the modality of these sublassical fragments of $\mathsf{S0.5}^0$ can be used as recovery operators.

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The connective $\blacktriangle$ represents a certain form of non-contingency. The modal approach to non-contingency was inaugurated by Montgomery & Routley (MONTGOMERY; ROUTLEY, 1966) and investigated by Humberstone (HUMBERSTONE, 1995) and Cresswell (CRESSWELL, 1988). By the semantic condition of $\blacktriangle$, $\blacktriangle\varphi$ receives the truth-value 1 if and only if φ receives 1 or 0 in all worlds y and z accessible to w.

$\mathsf{LFI1}^{\mathsf{S0.5}}$ to recover the inferences of S0.5 under certain non-contingency assumptions. Consider first the following definition:

Definition 4.2. *Let $\varphi \in For(\mathcal{L}^{\bullet}_{\Box\Diamond})$ be a $\mathsf{LFI1}^{\mathsf{S0.5}}$ formula. The modal degree of φ, $md(\varphi)$, is defined as follows:*

1. *if $\varphi = p$, then $md(p) = 0$;*
2. *if $\varphi = \neg\psi$, then $md(\neg\psi) = md(\psi)$;*
3. *if $\varphi = \psi \to \gamma$, then $md(\psi \to \gamma) = max(md(\psi), md(\gamma))$;*
4. *if $\varphi = \psi \vee \gamma$, then $md(\psi \vee \gamma) = max(md(\psi), md(\gamma))$;*
5. *if $\varphi = \bullet\psi$, then $md(\bullet\psi) = md(\psi)$;*
6. *if $\varphi = M\psi$, then $md(M\psi) = md(\psi) + 1$, for $M \in \{\Box, \Diamond, \blacksquare, \blacktriangle\}$;*

Theorem 4.3. *For every $\Gamma \subseteq For(\mathcal{L}^{\bullet}_{\Box\Diamond})$ such that $md(\gamma) \leq 1$, for every $\gamma \in \Gamma$, and for every $\varphi \in For(\mathcal{L}^{\bullet}_{\Box\Diamond})$ such that $md(\varphi) \leq 1$,*

$$\Gamma \vDash_{\mathsf{S0.5}} \varphi \text{ iff } \Gamma, \{\blacktriangle p_1, \ldots, \blacktriangle p_n\} \vDash_{\mathsf{LFI1}^{\mathsf{S0.5}}} \varphi \tag{4}$$

where $\{p_1, \ldots, p_n\}$ is the set of propositional variables which occur in $\Gamma \cup \{\varphi\}$.

The strategy to prove Theorem 4.3 is simple: whenever the hypothesis $\Gamma \cup \{\blacktriangle p_1, \ldots, \blacktriangle p_n\}$ hold, the valuations are forced to analyse the cases where the atomic formulas of the set $\{p_1, \ldots, p_n\}$ only receive classical values. Then, the result easily follows. Now, it is important to note that Theorem 4.3 does not generalize to any modal degree. Consider the following formula:

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$$\Box(\Box p \to (\neg\Box p \to \Box q)) \tag{5}$$

It easily verifiable that formula 5 is not valid in the models of $\mathsf{LFI1}^{S0.5}$. Since the value of modal formulas is arbitrary in non-normal worlds, the addition of the hypothesis $\blacktriangle p$ and $\blacktriangle q$ will not make any difference. So, Theorem 4.3 holds for formulas of modal degree lesser or equal 1.

Corollary 4.4. *The following items hold for* $\mathsf{LFI1}^{S0.5}$*:*

1. $\blacktriangle p, \blacktriangle q, p \vee q, \neg p \models q$;
2. $\blacktriangle p, \blacktriangle q, p, \neg p \models q$;
3. $\blacktriangle p, \blacktriangle q, p \to q, \neg q \models \neg p$;
4. $\blacktriangle p, \blacktriangle q, p \to q \models \neg q \to \neg p$.
5. $\models \blacktriangle\varphi \to \circ\varphi$

The results of this section shows that metatheoretical concepts can be used to recover classical inferences once we assume a classical metatheory for many-valued logics. Besides being recovery operators, they have a clear and well-grounded interpretation due to the results proved in (BEZERRA; VENTURI, 2022).

4.1 Classicality again

Now we will present a modality which captures the basic ideas of $\circ$. Then we will discuss what this modality has in common with the classicality connective $\circ$. In this case, such definition will not be so straightforward, because in this case we will depend on the expressiveness of $\mathsf{LFI1}^{S0.5}$ of defining a classical negation. Fortunately, LFIs in general are expressive enough to define a connective of classical negation. The definition runs as follows:

$$\sim\varphi \quad := \quad \circ\varphi \wedge \neg\varphi$$

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So their semantic interpretations are give by the following truth-tables:[15]

	$\sim$
1	0
$\frac{1}{2}$	0
0	1

Given the definition of classical negation, we define the modal operator of classicality as follows:

$$©\varphi \quad := \quad \sim\varphi \vee \blacksquare\varphi$$

Then, its semantical condition is stated as follows:

Definition 4.5. *Let $\mathcal{M} = \langle W, N, R, v\rangle$ be a M_{LFI1}-modal model and $w \in N$.*

$v_w(©\varphi) = 1$ *iff* $v_w(\varphi) = 0$ *or for all* $y \in W$ *such that* wRy*,* $v_y(\varphi) = 1$*; otherwise,* $v_w(©\varphi) = 0$

The connective © is close to the essence modal operator $\circ$ introduced by Marcos (MARCOS, 2005a) and widely investigated by Gilbert & Venturi (GILBERT; VENTURI, 2016).[16] In normal modal logics based on CPL, the essence operator is defined as $\varphi \to \Box\varphi$. In the present case, the $\varphi \to \psi$ is not equivalent to $\neg\varphi \vee \psi$. Then it was necessary to use disjunction and classical negation to define ©. It means that the modal study of the modality of classicality is not possible in every MVL because not every many-valued logic is capable to define classical negation.

Similarly to the case of $\blacktriangle$ we now state a semantic version of DAT using the connective ©.

[15]Note that the reading of classical negation sound much more natural when $\circ$ is interpreted as classicality.

[16]The fact that the LFI1's $\circ$ and the essence operator have the same symbol is not accidental. In these aforementioned papers, the authors claim that the essence operator can be interpreted as expressing a form of consistency.

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Theorem 4.6. *For every* $\Gamma \subseteq For(\mathcal{L}^{\bullet}_{\Box\Diamond})$ *such that* $md(\gamma) \leq 1$*, for every* $\gamma \in \Gamma$*, and for every* $\varphi \in For(\mathcal{L}^{\bullet}_{\Box\Diamond})$ *such that* $md(\varphi) \leq 1$*,*

$$\Gamma \models_{\mathsf{S0.5}} \varphi \text{ iff } \Gamma, \{©p_1, \ldots, ©p_n\} \models_{\mathsf{LFI1}^{\mathsf{S0.5}}} \varphi \tag{6}$$

where $\{p_1, \ldots, p_n\}$ *is the set of propositional variables which occur in* $\Gamma \cup \{\varphi\}$.

The strategy of the proof of Theorem 4.6 is the same of Theorem 4.3. As one can check, the operator satisfies the following schemas:

Proposition 4.7. *The following items hold in* $\mathsf{LFI1}^{\mathsf{S0.5}}$*:*

1. $©\bot$;
2. $©\varphi \to (\varphi \to (\neg\varphi \to \psi))$;
3. $(©\varphi \wedge ©\psi) \to ©(\varphi \wedge \psi)$.

We can see that © satisfies at least the minimal properties of $\circ$ when we have formulas with modal degree lesser or equal one. They cease to coincide at least when the modal degree is greater than one. In this case, there is no problem since the iteration of $\circ$ does not seem to conflict with the classicality interpretation. But we do not see a conflict here. So, in this case, the classicality interpretation of $\circ$ seems to be more adequate. As we said before, $\circ$ clearly echoes a metatheoretical notion.

5 Conclusion

In this work, we showed that the modalities of $\mathsf{LFI1}^{\mathsf{S0.5}}$ can be used as recovery operators, in the sense that they recover the inferences of the classical S0.5. Their metatheoretical meaning is precise, since we can show that they capture well justified notions of validity and consistency. In the context of provability logics, Verbrugge (VERBRUGGE, 2017) argues that the provability interpretation of modal logics fulfils Quine's challenge to modal logics (QUINE, 1966). Since the modalities investigated here also refer to metatheoretical notions, we can say that the present study also fulfils his challenge. So the extension of the

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present work to stronger modalities that also capture stronger notions of logical validities looks promising.

One could argue that the problems raised in Subsection 2.1 are just a verbal dispute. We do not think that is the case for two reasons. First, the use of such operators became widespread, mainly in philosophical discussions about about the relation between classical and non-classical logics. Second, by the proper fact that LFIs are understood as paraconsistent logics which internalize a metatheoretical notion of consistency, we do not think that this question is trivial. If it were the case that the truth-functional connective $\circ$ did not capture any metatheoretical notion, the LFIs would not represent any conceptual gain in relation to the other paraconsistent logics that do not have similar connectives to $\circ$.

A Proof systems for $\mathsf{LFI1}^{\mathsf{S0.5}}$

Now we will present a labelled tableaux proof system for $\mathsf{LFI1}^{\mathsf{S0.5}}$, which is a generalization of Carnielli's method (CARNIELLI, W. A., 1987) to modal many-valued logics presented in (BEZERRA, 2021).

Definition A.1. *Let φ be a formula and $[t]$ be a label, for $t \in \{1, \frac{1}{2}, 0\}$. A* signed formula *has the form $[t]\varphi$.*

Definition A.2. *Let $[t]\varphi, i$ be a signed formula. Given $[t]\varphi, i$, we construct a* tree *(a* tableau*) for $[t]\varphi, i$ as follows:*

(i) *$[t]\varphi, i$ is the* root*/initial* node *of the tree;*

(ii) *We expand the root of the tree into* branches *b by applying the rule for $[t]\varphi, i$. Every such b contains signed formulas resulting from the application of the rule for $[t]\varphi, i$, and possibly $k\mathfrak{r}l$, where $\mathfrak{r}$ is a rule and $k, l \in \mathbb{N}$, in the case that φ is a modal formula.*

(iii) *The* endpoints *of the tree are nodes which contains formulas for which there is no rule to be applied.*

The Definition A.2 can be extended for sets of formulas Γ in the obvious way.

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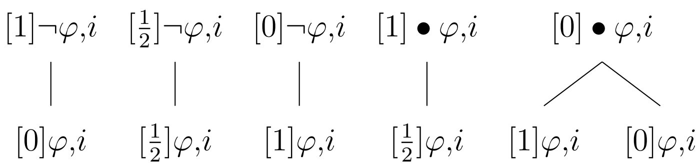
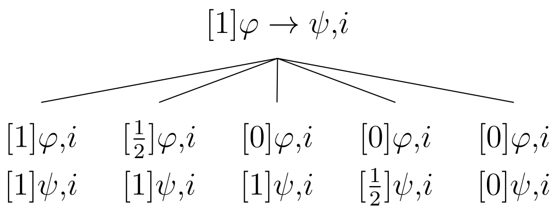
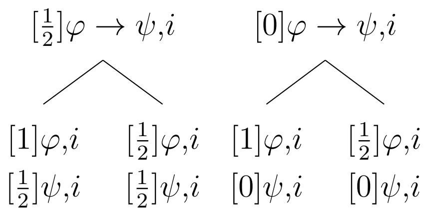
Definition A.3. *Let $\mathcal{T}$ be a tableau and b be a branch of $\mathcal{T}$. We say that b is* complete *if every rule which can be applied is applied. $\mathcal{T}$ is complete if its branches are complete.*

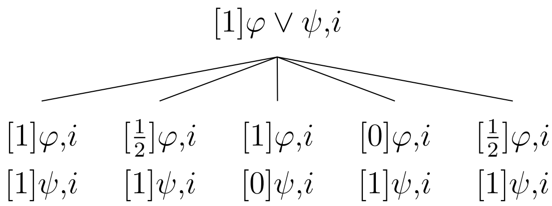
Definition A.4. *Let $\mathcal{T}$ be a tableau and b be a branch of $\mathcal{T}$. We say that b* closes *(i) if there is a formula φ such that $[t]\varphi, j$ and $[t']\varphi, j$ with $t \neq t'$, such that $t, t' \in \{1, \frac{1}{2}, 0\}$, occurring in b; (ii) if $[\frac{1}{2}]\Box\varphi, 0$ or $[\frac{1}{2}]\Diamond\varphi, 0$ occur in b.*

The rules of the connectives are given bellow:









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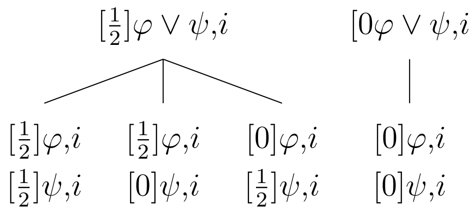

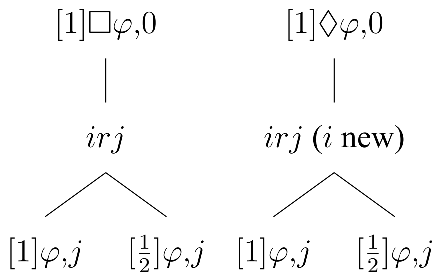

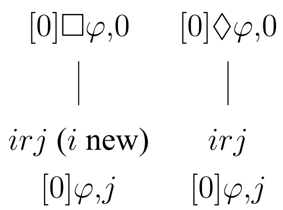

The rule r obbeys the following constraint:

$$\frac{\cdot}{iri}\,(\text{rule } \rho)$$

The notion of proof is defined as follows.

Definition A.5. $\Sigma \vdash_{\mathsf{LFI1}^{\mathsf{S0.5}}} \varphi$ *if there is a closed tableau* $\mathcal{T}$*'s such that:*

1. *For each* $\sigma_i \in \Sigma$, $[t]\sigma_i, 0$, *where* $t \in \{1, \frac{1}{2}\}$;
2. *If* $\sigma_i \in \Sigma$ *and* $\sigma_i = \Box\psi$, *then* $[1]\Box\psi, 0$;

3. *If* $\sigma_i \in \Sigma$ *and* $\sigma_i = \Diamond\psi$, *then* $[1]\Diamond\psi, 0$;

4. $[0]\varphi, 0$.

The characterization results for the proof system for $\mathsf{LFI1}^{\mathsf{S0.5}}$ are direct consequences of the results proved in (BEZERRA, 2021).

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